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The Cauchy problem for nonlinear quadratic interactions of the Schrödinger type in one dimensional space

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In this work, we study the well-posedness of the Cauchy problem associated with the coupled Schrödinger equations with quadratic nonlinearities, which appears modeling problems in nonlinear optics. We obtain the local well-posedness for data in Sobolev spaces with low regularity. To obtain the local theory, we prove new bilinear estimates for the coupling terms of the system in the continuous case. Concerning global results, in the continuous case, we establish the global well-posedness in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, for some negatives indexes s. The proof of our global result uses the **I**-method introduced by Colliander *et al*. *Published by AIP Publishing.* https://doi.org/10.1063/1.5045337

I. INTRODUCTION

This work is dedicated to the study of the Cauchy problem for a system that appears modeling some problems in the context of nonlinear optics. More precisely, we will study the following mathematical model:

$$\begin{cases} i\partial_t u(x,t) + p\partial_x^2 u(x,t) - \theta u(x,t) + \bar{u}(x,t)v(x,t) = 0, & x \in \mathbb{R},\ t \geq 0, \\ i\sigma\partial_t v(x,t) + q\partial_x^2 v(x,t) - \alpha v(x,t) + \frac{a}{2}u^2(x,t) = 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases} \tag{1}$$

where u and v are complex valued functions and α, θ, and $a := 1/\sigma$ are real numbers representing physical parameters of the system, where $\sigma > 0$ and $p, q = \pm 1$. The model (1) is given by the nonlinear coupling of two dispersive equations of Schrödinger type through the quadratic terms

$$N_1(u,v) = \overline{u} \cdot v \ \text{ and } N_2(u,v) = \frac{1}{2} u \cdot v. \tag{2}$$

Physically, according to the article,[15] the complex functions u and v represent amplitude packets of the first and second harmonic of an optical wave, respectively. The values of p and q may be 1 or -1, depending on the signals provided between the scattering/diffraction ratios and the positive constant σ measures the scaling/diffraction magnitude indices. In recent years, interest in nonlinear properties of optical materials has attracted attention of physicists and mathematicians. Many research studies suggest that by exploring the nonlinear reaction of the matter, the bit-rate capacity of optical fibers can be considerably increased and in consequence an improvement in the speed and economy of data transmission and manipulation. Particularly in non-centrosymmetric materials, those having no inversion symmetry at the molecular level, the nonlinear effects of lower order give rise to second order susceptibility, which means that the nonlinear response to the electric field is quadratic; see, for instance, Refs. 12 and 9.

Another application for system (1) is related to the Raman amplification in a plasma. The study of laser-plasma interactions is an active area of interest. The main goal is to simulate nuclear fusion in a laboratory. In order to simulate numerically these experiments, we need some accurate models. The kinetic ones are the most relevant but very difficult to deal with practical computations. The fluids

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ones like the bifluid Euler–Maxwell system seem more convenient but still inoperative in practice because of the high frequency motion and the small wavelength involved in the problem. This is why we need some intermediate models that are reliable from a numerical viewpoint.[3]

In the mathematical context, Hayashi, Ozawa, and Tanaka in Ref. 11 obtained local well-posedness for the Cauchy problem (1) on the spaces $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ for $n \leq 4$ and $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ for $n \leq 6$. In Ref. 14, the time decay estimates of small solutions to the systems under the mass resonance condition in 2-dimensional space were revised. The authors also showed the existence of wave operators and modified wave operators of the systems under some mass conditions in n-dimensional space, where $n \geq 2$, and showed the existence of scattering operators and finite time blow-up of the solutions for the systems in higher dimensional spaces.

Regarding to qualitative properties of Cauchy problem solutions (1), we know that in the case where $p = q = 1$ the system was studied by Linares and Angulo in Ref. 1 for initial data u_0, v_0 in the same periodic Sobolev space $H^s(\mathbb{T})$. More precisely, they obtained local well-posedness results in $H^s(\mathbb{T}) \times H^s(\mathbb{T})$ for all $s \geq 0$ and obtained global well-posedness in the space $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ using the conservation of the mass by the flow of the system, that is, the following conservation law:

$$E(u(t), v(t)) = \int_{-\infty}^{+\infty} (|u|^2 + 2\sigma |v|^2)dx = E(u_0, v_0). \tag{3}$$

Remark 1. The authors also observed in Comment 2.3 of Ref. 1 *that results can be obtained for data with lower regularity when σ is different from 1, including well-posedness in $H^s_{per} \times H^s_{per}$ for $s > -1/2$. Furthermore, in the same work, stability and instability results were established for certain classes of periodic pulses. Another work devoted to the study of the existence and stability of wave type pulses for this model is due to Yew (see* Ref. 17*).*

The techniques used in Ref. 1 to obtain the results of local well-posedness follow the ideas in Ref. 13, developed by Kenig, Ponce, and Vega, where the initial value problem for a Schrödinger equation with quadratic nonlinearities in both periodic and continuous domains is studied. More precisely, they considered the following initial value problem:

$$\begin{cases} iu_t + \partial_x^2 u = N_j(u, \bar{u}), & x \in \mathbb{R} \text{ or } x \in \mathbb{T}, t \geq 0, \\ u(x, 0) = u_0(x), \end{cases} \tag{4}$$

where $N_1(u, \bar{u}) = u\bar{u}$, $N_2(u, \bar{u}) = u^2$, and $N_3(u, \bar{u}) = \bar{u}^2$. The authors considered initial data in the Sobolev space H^s. In the continuous case, they proved local well-posedness for $s > -1/4$ in the case $j = 1$ and for $s > -3/4$ in the cases $j = 2, 3$. In the periodic case, local well-posedness was obtained for $s \geq 0$ when $j = 1$ and for $s > -1/2$ when $j = 2, 3$. To prove the local theory, they used the Fourier restriction method, known in the literature, as $X^{s,b}$-spaces and introduced by Bourgain in Ref. 2. In this functional space, sharp bilinear estimates were proved. These estimates combined with the Banach Fixed Point Theorem applied to the integral operator associated with (1) allowed us to obtain the desired local solutions. The lack of a conservation law for (4) does not allow us to get global results in some space, as usual.

We note that the results given in Ref. 13 can be applied to system (1) in the case where $\sigma = 1$. In this situation, it is not difficult to obtain the local well-posedness in $H^s \times H^s$ for $s > -1/4$. However, a natural question arises:

> *What would be the scenery of the local and global well-posedness of system* (1) *when $\sigma \neq 1$ and for initial data in Sobolev spaces, not necessarily with the same regularity?*

In this work, we consider the Cauchy problem (1) with any $\sigma > 0$ and initial data (u_0, v_0) belonging to Sobolev spaces $H^\kappa(\mathbb{R}) \times H^s(\mathbb{R})$ to answer the previous question. As far as we know, the local well-posedness for system (1) in low regularity is unknown.

We will follow the ideas developed by Corcho and Matheus in Ref. 8, where they treated the Schrödinger-Debye system, modelled by

$$\begin{cases} iu_t + \frac{1}{2}\partial_x^2 u = uv, & x \in \mathbb{R},\ t \geq 0, \\ \mu v_t + v = \pm |u|^2, & \mu > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & \end{cases} \tag{5}$$

which also has quadratic type nonlinearities and the authors developed a local and global theory in Sobolev spaces with different regularities. They used the method also based on obtaining sharp bilinear estimates for the coupling terms in suitable Bourgain spaces as well as the use of fixed point techniques.

Moreover, in the same work, global results were obtained by using a technique known as the **I**-method which was first implemented by Colliander *et al.* in Ref. 4.

Before enunciating the main results, we have given the following definition.

Definition 1. Given $\sigma > 0$, *we say that the Sobolev index pair* (κ, s) *verifies the hypotheses* H_σ *if it satisfies one of the following conditions*:

(a) $|\kappa| - 1/2 \leq s < \min\{\kappa + 1/2, 2\kappa + 1/2\}$ *for* $0 < \sigma < 2$;
(b) $\kappa = s \geq 0$ *for* $\sigma = 2$;
(c) $|\kappa| - 1 \leq s < \min\{\kappa + 1, 2\kappa + 1\}$ *for* $\sigma > 2$.

We denote

$$\mathcal{W}_\sigma := \left\{(\kappa, s) \in \mathbb{R}^2;\ (\kappa, s) \text{ verify the hypothesis } H_\sigma\right\}. \tag{6}$$

Throughout the paper, we fix a cutoff function ψ in C_0^∞ such that $0 \leq \psi(t) \leq 1$,

$$\psi(t) = \begin{cases} 1, & \text{if} \quad |t| \leq 1 \\ 0, & \text{if} \quad |t| \geq 2 \end{cases} \tag{7}$$

and $\psi_T(t) = \psi\left(\frac{t}{T}\right)$.

Our main local well-posedness result is the following statement.

Theorem 1. *For any* $\sigma > 0$ *and* $(u_0, v_0) \in H^\kappa \times H^s$ *where the Sobolev index pair* (κ, s) *verifying the hypothesis* H_σ, *there exist a positive time* $T = T(\|u_0\|_{H^\kappa}, \|v_0\|_{H^s}, \sigma)$ *and an unique solution* $(u(t), v(t))$ *for the initial value problem* (1), *satisfying*

$$\psi_T(t)u \in X^{\kappa,\frac{1}{2}+} \quad \text{and} \quad \psi_T(t)v \in X^{\kappa,\frac{1}{2}+}_{1/\sigma}, \tag{8}$$

$$u \in C([0,T]; H^\kappa(\mathbb{R})) \quad \text{and} \quad v \in C([0,T]; H^s(\mathbb{R})). \tag{9}$$

Moreover, the map $(u_0, v_0) \longmapsto (u(t), v(t))$ *is locally Lipschitz from* $H^\kappa(\mathbb{R}) \times H^s(\mathbb{R})$ *into* $C([0,T]; H^\kappa(\mathbb{R}) \times H^s(\mathbb{R}))$.

Concerning global well-posedness, we have the following result.

Theorem 2. *In the following cases*:

- $\sigma = 2$ *and* $s = 0$;
- $\sigma > 2$ *and* $s \geq -1/2$;
- $0 < \sigma < 2$ *and* $s \geq -1/4$.

The Cauchy problem associated with system (1) *is globally well-posed,* i.e., *there exists a unique solution for any* $T > 0$ *with initial condition* $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$.

Now we describe the structure of our work. Section II is devoted to summarize some preliminary results. In Sec. III, we will develop a local theory in Bourgain spaces, following closely the techniques used in Refs. 13 and 8, where for each positive σ we obtain quite general results in Sobolev spaces

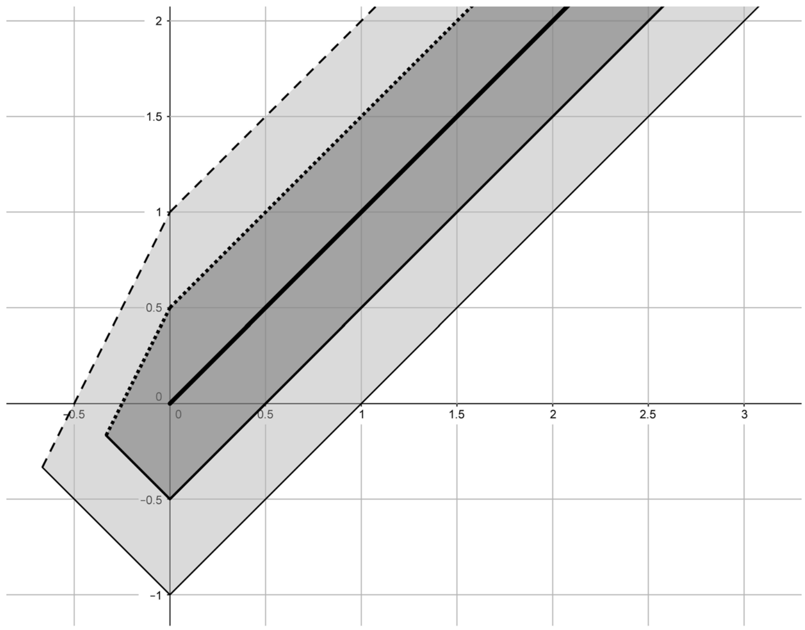

FIG. 1. Region $\mathcal{W}_\sigma$.

with regularities out of the diagonal case $\kappa = s$. Specifically, we will prove local well-posedness for data $(u_0, v_0) \in H^\kappa \times H^s$ with indices $(\kappa, s) \in \mathcal{W}_\sigma$ (see Fig. 1).

Finally, in Sec. IV, we will use the I-method to extend globally the local solutions obtained for data in $H^s \times H^s$ with values of s for some negatives. More precisely, we have regularity $-\frac{1}{4} \leq s \leq 0$ when $0 < \sigma < 2$ and $-\frac{1}{2} \leq s \leq 0$ when $\sigma > 2$. At this point, the use of a refined Strichartz-type estimate in Bourgain's spaces for the Schrödinger equation will be crucial. For details, the reader can see Ref. 5.

II. PRELIMINARY RESULTS

We consider the equation of the form

$$i\partial_t \omega - \phi(-i\partial_x)\omega = F(\omega), \tag{10}$$

where ϕ is a measurable real-valued function and F is a nonlinear function.

The Cauchy problem for (10) with initial data $\omega(0) = \omega_0$ is rewritten as the following integral equation:

$$\omega(t) = W_\phi(t)\omega_0 - i\int_0^t W_\phi(t-t')F(\omega(t'))dt', \tag{11}$$

where $W_\phi(t) = e^{-it\phi(-i\partial_x)}$ is the group that solves the linear part of (10).

Let $X^{s,b}(\phi)$ be the completion of $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm

$$\begin{aligned} \|f\|_{X^{s,b}(\phi)} &:= \left\| W_\phi(-t)f \right\|_{H_t^b(\mathbb{R},H_x^s)} \\ &= \left\| \langle\xi\rangle^s \langle\tau\rangle^b \mathcal{F}\left(e^{it\phi(-i\partial_x)}f\right)(\tau,\xi)\right\|_{L_\tau^2 L_\xi^2} \\ &= \left\| \langle\xi\rangle^s \langle\tau+\phi(\xi)\rangle^b \widehat{f}(\tau,\xi)\right\|_{L_\tau^2 L_\xi^2}. \end{aligned} \tag{12}$$

The following lemma was proved while establishing the local well-posedness of the Zakharov system by Ginibre, Tsutsumi, and Velo in Ref. 10.

Lemma 1. Let $-\frac{1}{2} < b' \leq 0 \leq b \leq b' + 1$, ψ *be a cutoff function, and* $T \in [0, 1]$. *Then for* $F \in X^{s,b'}(\phi)$, *we have*

$$\left\| \psi_1(t) W_\phi(t)\omega_0 \right\|_{X^{s,b}(\phi)} \leq C\|\omega_0\|_{H^s}, \tag{13}$$

$$\left\| \psi_T(t) \int_0^t W_\phi(t-t')F(\omega(t'))dt' \right\|_{X^{s,b}(\phi)} \leq CT^{1+b'-b}\|F\|_{X^{s,b'}(\phi)}. \tag{14}$$

Proof. See Lemma 2.1 in Ref. 10. □

In our case, we shall use the space $X^{s,b}(\phi)$ for the phase functions $\phi_1(\xi) = \xi^2$ and $\phi_a(\xi) = a\xi^2$. Indeed we can rewrite system (1) in the form

$$\begin{cases} i\partial_t u - \phi_1(-i\partial_x)u - \theta u + \bar{u}v = 0, \\ i\partial_t v - \phi_a(-i\partial_x)v - \alpha v + \frac{a}{2}u^2 = 0, \quad a > 0. \end{cases} \tag{15}$$

Then we have

$$X^{\kappa,b}(\phi_1) = X^{\kappa,b}, \quad W_{\phi_1} = e^{it\partial_x^2}$$

and

$$X^{s,b}(\phi_a) = X_a^{s,b}, \quad W_{\phi_a} = e^{iat\partial_x^2}.$$

We finish this section with the following elementary integral estimates which will be used to estimate the nonlinear terms in Sec. III.

Lemma 2. Let $p, q > 0$, *for* $r = min\{p, q\}$ *with* $p + q > 1 + r$, *there exists* $C > 0$ *such that*

$$\int_{\mathbb{R}} \frac{dx}{\langle x-\alpha\rangle^p \, \langle x-\beta\rangle^q} \leq \frac{C}{\langle \alpha-\beta\rangle^r}. \tag{16}$$

Moreover, for $q > \frac{1}{2}$,

$$\int_{\mathbb{R}} \frac{dx}{\langle \alpha_0 + \alpha_1 x + x^2\rangle^q} \leq C \; \textit{for all } \alpha_0, \alpha_1 \in \mathbb{R}. \tag{17}$$

Proof. See Lemma 2.3 in Ref. 13. □

III. BILINEAR ESTIMATES FOR THE COUPLING TERMS

The main results in the section are the following propositions which present the bilinear estimates for different values of $\sigma > 0$. Each case leads us to different restrictions on the Sobolev indices s and κ.

A. Bilinear estimates for $\sigma > 2$

Next we prove a new bilinear estimate when $\sigma > 2$ ($\sigma = 1/a$).

Proposition 1. Let $0 < a < \frac{1}{2}$ *(equivalently* $\sigma > 2$*),* $u \in X^{\kappa,b}$, *and* $v \in X_a^{s,b}$ *with* $1/2 < b < 3/4$, $1/4 < d < 1/2$, *and* $|\kappa| - s \leq 1$, *then the bilinear estimate holds*

$$\|\overline{u} \cdot v\|_{X^{\kappa,-d}} \leq C\|u\|_{X^{\kappa,b}} \cdot \|v\|_{X^{s,b}}. \tag{18}$$

The second result is the following

Proposition 2. Let $0 < a < \frac{1}{2}$ *(equivalently* $\sigma > 2$*) and* $u, \tilde{u} \in X^{\kappa,b}$ *with* $1/2 < b < 3/4$, $1/4 < d < 1/2$ *and* $s < \kappa + 1$ *if* $\kappa \geq 0$ *and* $s < 2\kappa + 1$ *if* $\kappa < 0$ *then, the following estimate holds*

$$\|u \cdot \tilde{u}\|_{X_a^{s,-d}} \leq C\|u\|_{X^{\kappa,b}} \cdot \|\tilde{u}\|_{X^{\kappa,b}}. \tag{19}$$

Proof of the Proposition 1. We define

$$f(\xi,\tau)=\langle\tau-\xi^2\rangle^b\langle\xi\rangle^\kappa\widehat{\overline{u}}(\xi,\tau) \text{ and } g(\xi,\tau)=\langle\tau+a\xi^2\rangle^b\langle\xi\rangle^s\widehat{v}(\xi,\tau).$$

Therefore, $\|f\|_{L^2_{\xi,\tau}}=\|u\|_{X^{\kappa,b}}$ and $\|g\|_{L^2_{\xi,\tau}}=\|v\|_{X_a^{s,b}}$.
It follows that

$$\begin{aligned}\|\overline{u}\cdot v\|_{X^{\kappa,-d}}&=\left\|\langle\tau+\xi^2\rangle^{-d}\langle\xi\rangle^\kappa\,\widehat{\overline{u}\cdot v}(\xi,\tau)\right\|_{L^2_{\xi,\tau}}\\&=\sup_{\|\varphi\|_{L^2_{\xi,\tau}}\le 1}\left|\int_{\mathbb{R}^4}\frac{\langle\xi\rangle^\kappa\langle\xi_1\rangle^{-\kappa}\langle\xi_2\rangle^{-s}}{\langle\tau+\xi^2\rangle^d\langle\tau_1-\xi_1^2\rangle^b\langle\tau_2+a\xi_2\rangle^b}f(\xi_1,\tau_1)g(\xi_2,\tau_2)\varphi(\xi,\tau)d\xi_2d\tau_2d\xi d\tau\right|.\end{aligned}$$

We use the following notation:

$$\begin{cases}\tau=\tau_1+\tau_2 \quad \xi=\xi_1+\xi_2\\ \omega=\tau+\xi^2,\ \omega_1=\tau_1-\xi_1^2,\ \omega_2=\tau_2+a\xi_2^2\end{cases}\tag{20}$$

and we define

$$W(f,g,\varphi)=\int_{\mathbb{R}^4}\frac{\langle\xi\rangle^\kappa\langle\xi_1\rangle^{-\kappa}\langle\xi_2\rangle^{-s}}{\langle\omega\rangle^d\langle\omega_1\rangle^b\langle\omega_2\rangle^b}f(\xi_1,\tau_1)g(\xi_2,\tau_2)\varphi(\xi,\tau)d\xi_2d\tau_2d\xi d\tau.$$

Now it is suffices to prove that

$$|W(f,g,\varphi)|\le c\ \|f\|_{L^2}\cdot\|g\|_{L^2}\cdot\|\varphi\|_{L^2}.$$

Consider $\mathbb{R}^4\subset\mathcal{R}_1\cup\mathcal{R}_2\cup\mathcal{R}_3$, where $\mathcal{R}_j\subset\mathbb{R}^4$ for $j\in\{1,2,3\}$. We write

$$W_j=(f,g,\varphi)=\int_{\mathcal{R}_j}\frac{\langle\xi\rangle^\kappa\langle\xi_1\rangle^{-\kappa}\langle\xi_2\rangle^{-s}}{\langle\omega\rangle^d\langle\omega_1\rangle^b\langle\omega_2\rangle^b}f(\xi_1,\tau_1)g(\xi_2,\tau_2)\varphi(\xi,\tau)d\xi_2d\tau_2d\xi d\tau$$

and observe that $|W|\le|W_1|+|W_2|+|W_3|$.

We estimate each case separately. Using the Cauchy-Schwarz and Hölder inequalities and Fubini's Theorem, we obtain

$$\begin{aligned}|W_1|^2&=\left|\int_{\mathcal{R}_1}\frac{\langle\xi\rangle^\kappa\langle\xi_1\rangle^{-\kappa}\langle\xi_2\rangle^{-s}}{\langle\omega\rangle^d\langle\omega_1\rangle^b\langle\omega_2\rangle^b}f(\xi_1,\tau_1)g(\xi_2,\tau_2)\varphi(\xi,\tau)d\xi_2d\tau_2d\xi d\tau\right|^2\\&\le\|f\|_{L^2}^2\|g\|_{L^2}^2\|\varphi\|_{L^2}^2\left\|\frac{\langle\xi\rangle^{2\kappa}}{\langle\omega\rangle^{2d}}\left(\int_{\mathbb{R}^2}\frac{\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2s}\chi_{\mathcal{R}_1}d\xi_2d\tau_2}{\langle\omega_1\rangle^{2b}\langle\omega_2\rangle^{2b}}\right)\right\|_{L^\infty_{\xi,\tau}}.\end{aligned}$$

Similarly, we have

$$|W_2|^2\le\|f\|_{L^2}^2\|g\|_{L^2}^2\|\varphi\|_{L^2}^2\left\|\frac{\langle\xi_2\rangle^{2s}}{\langle\omega_2\rangle^{2b}}\left(\int_{\mathbb{R}^2}\frac{\langle\xi_1\rangle^{-2\kappa}\langle\xi\rangle^{2\kappa}}{\langle\omega_1\rangle^{2b}\langle\omega\rangle^{2d}}\chi_{\mathcal{R}_2}d\xi d\tau\right)\right\|_{L^\infty_{\xi_2,\tau_2}}$$

and

$$|W_3|^2\le\|f\|_{L^2}^2\|g\|_{L^2}^2\|\varphi\|_{L^2}^2\left\|\frac{\langle\xi_1\rangle^{-2\kappa}}{\langle\omega_1\rangle^{2b}}\left(\int_{\mathbb{R}^2}\frac{\langle\xi\rangle^{2\kappa}\langle\xi_2\rangle^{-2s}}{\langle\omega\rangle^{2d}\langle\omega_2\rangle^{2b}}\chi_{\mathcal{R}_3}d\xi_2d\tau_2\right)\right\|_{L^\infty_{\xi_1,\tau_1}}.$$

Using Lemma 17 and the fact $\langle\xi\rangle^{2\kappa}\langle\xi_1\rangle^{-2\kappa}\le\langle\xi_2\rangle^{2|\kappa|}$, we get the following inequalities:

$$\frac{\langle\xi\rangle^{2\kappa}}{\langle\omega\rangle^{2d}}\int_{\mathbb{R}^2}\frac{\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2s}\chi_{\mathcal{R}_1}}{\langle\omega_1\rangle^{2b}\langle\tau_2+a\xi_2^2\rangle^{2b}}d\xi_2d\tau_2\le\underbrace{\frac{1}{\langle\omega\rangle^{2d}}\int_{\mathbb{R}}\frac{\langle\xi_2\rangle^{-2s+2|\kappa|}\chi_{\mathcal{R}_1}}{\langle\tau-(a-1)\xi_2^2-2\xi\xi_2+\xi^2\rangle^{2b}}d\xi_2}_{J_1},$$

$$\frac{\langle\xi_2\rangle^{2s}}{\langle\omega_2\rangle^{2b}}\int_{\mathbb{R}^2}\frac{\langle\xi_1\rangle^{-2\kappa}\langle\xi\rangle^{2\kappa}\chi_{\mathcal{R}_2}}{\langle\omega_1\rangle^{2b}\langle\omega\rangle^{2d}}d\xi d\tau \le \underbrace{\frac{1}{\langle\tau_2+a\xi_2^2\rangle^{2b}}\int_{\mathbb{R}}\frac{\langle\xi_2\rangle^{-2s+2|\kappa|}\chi_{\mathcal{R}_2}}{\langle\tau_2+2\xi^2+\xi_2^2-2\xi\xi_2\rangle^{2d}}d\xi}_{J_2},$$

$$\frac{\langle\xi_1\rangle^{-2\kappa}}{\langle\omega_1\rangle^{2b}}\int_{\mathbb{R}^2}\frac{\langle\xi\rangle^{2\kappa}\langle\xi_2\rangle^{-2s}\chi_{\mathcal{R}_3}}{\langle\omega\rangle^{2d}\langle\omega_2\rangle^{2b}}d\xi_2 d\tau_2 \le \underbrace{\frac{1}{\langle\tau_1-\xi_1^2\rangle^{2b}}\int_{\mathbb{R}}\frac{\langle\xi_2\rangle^{-2s+2|\kappa|}\chi_{\mathcal{R}_3}}{\langle\tau_1-a\xi_2^2+\xi^2\rangle^{2d}}d\xi_2}_{J_3}.$$

It is enough to show that the functionals J_1, J_2, and J_3, defined below, are bounded

$$J_1=\frac{1}{\langle\tau+\xi^2\rangle^{2d}}\int_{\mathbb{R}}\frac{\langle\xi_2\rangle^{-2s+2|\kappa|}\chi_{\mathcal{R}_1}}{\langle\tau-(a-1)\xi_2^2-2\xi\xi_2+\xi^2\rangle^{2b}}d\xi_2, \tag{21}$$

$$J_2=\frac{1}{\langle\tau_2+a\xi_2^2\rangle^{2b}}\int_{\mathbb{R}}\frac{\langle\xi_2\rangle^{-2s+2|\kappa|}\chi_{\mathcal{R}_2}}{\langle\tau_2+2\xi^2+\xi_2^2-2\xi\xi_2\rangle^{2d}}d\xi, \tag{22}$$

$$J_3=\frac{1}{\langle\tau_1-\xi_1^2\rangle^{2b}}\int_{\mathbb{R}}\frac{\langle\xi_2\rangle^{-2s+2|\kappa|}\chi_{\mathcal{R}_3}}{\langle\tau_1-a\xi_2^2+\xi^2\rangle^{2d}}d\xi_2. \tag{23}$$

In order to do so, we start by discussing the dispersion of relations. Note that

$$\begin{aligned}|\omega-\omega_1-\omega_2| &= |\xi^2+\xi_1^2-a\xi_2^2|\\ &\ge |1-a|(\xi^2+\xi_1^2)-2a|\xi\xi_1|, \quad \text{suppose } 0<a<\frac{1}{2}\\ &\ge (1-a)(\xi^2+\xi_1^2)-a(\xi^2+\xi_1^2)=(1-2a)(\xi^2+\xi_1^2).\end{aligned}$$

It follows that

$$3\max\{|\omega|,|\omega_1|,|\omega_2|\}\ge(1-2a)\max\{\xi^2,\xi_1^2\}\ge\frac{1-2a}{4}\xi_2^2.$$

Suppose that $|\xi_2|\ge 1$, then we have

$$\frac{1}{\max\{|\omega|,|\omega_1|,|\omega_2|\}}\le\frac{c}{|\xi_2|^2}.$$

Now, we define $\mathcal{R}_j$,

$$\mathcal{R}_1=\Big\{|\xi_2|\ge1, |\omega|=\max\{|\omega|,|\omega_1|,|\omega_2|\}\Big\}\cup\Big\{|\xi_2|\le1\Big\}\subset\mathbb{R}^4_{\xi,\tau,\xi_2,\tau_2}, \tag{24}$$

$$\mathcal{R}_2=\Big\{|\xi_2|\ge1, |\omega_1|=\max\{|\omega|,|\omega_1|,|\omega_2|\}\Big\}\subset\mathbb{R}^4_{\xi,\tau,\xi_2,\tau_2}, \tag{25}$$

$$\mathcal{R}_3=\Big\{|\xi_2|\ge1, |\tau_2+a\xi^2|=\max\{|\omega|,|\omega_1|,|\omega_2|\}\Big\}\subset\mathbb{R}^4_{\xi,\tau,\xi_2,\tau_2}. \tag{26}$$

Let us prove that J_1 is bounded. Indeed, if $|\xi_2|\le 1$, then J_1 is equivalent to

$$\frac{1}{\langle\omega\rangle^{2d}}\int_{|\xi_2|\le1}\frac{1}{\langle\tau-(a-1)\xi_2^2-2\xi\xi_2+\xi^2\rangle^{2b}}d\xi_2\le c.$$

If $|\xi_2|\ge 1$, then J_1 is bounded by

$$\int_{|\xi_2|\ge1}\frac{\langle\xi_2\rangle^{-2s+2|\kappa|+4d}\chi_{\mathcal{R}_1}}{\langle\tau-(a-1)\xi_2^2-2\xi\xi_2+\xi^2\rangle^{2b}}d\xi_2.$$

Note that J_1 is bounded, when $|\kappa|-s\le 2d<1$ because $b>1/2$.

To prove that J_2 is bounded, it is suffices to note that the integral below is higher than J_2 and that converges since $|\kappa|-s\le 2b$ and that $2d>1/2$, that is, $b<3/4$.

$$\int_{\mathbb{R}}\frac{\langle\xi_2\rangle^{-2s+2|\kappa|-4b}\chi_{\mathcal{R}_2}}{\langle\tau_2+2\xi^2+\xi_2^2-2\xi\xi_2\rangle^{2d}}d\xi.$$

Analogously, a similar way, we can prove that J_3 is bounded, by using that $|\kappa| - s \leq 2b$ and $b < 3/4$. □

Now we prove that the second non-linear term of the system is bounded.

Proof of the Proposition 2. Analogous to the previous proposition, the estimate (19) is equivalent to prove that the functionals J_4, J_5, and J_6, defined below, are bounded

$$J_4 = \frac{1}{\langle \lambda \rangle^{2d}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \chi_{\mathcal{S}_1}}{\langle \tau + \xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2, \tag{27}$$

$$J_5 = \frac{1}{\langle \lambda_2 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \chi_{\mathcal{S}_2}}{\langle \tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2 \rangle^{2d}} d\xi, \tag{28}$$

$$J_6 = \frac{1}{\langle \lambda_1 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \chi_{\mathcal{S}_3}}{\langle \tau_1 + a\xi^2 + \xi_2^2 \rangle^{2d}} d\xi_2, \tag{29}$$

where $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 = \mathbb{R}^4$ with $\mathcal{S}_j$ being measurable.

Note that

$$\begin{aligned}
|\lambda - \lambda_1 - \lambda_2| &= |a\xi^2 - \xi_1^2 - \xi_2^2| \\
&\geq |1-a|(\xi_1^2 + \xi_2^2) - 2a|\xi_1\xi_2|, \quad \text{suppose } 0 < a < \frac{1}{2} \\
&\geq (1-a)(\xi_1^2 + \xi_2^2) - a(\xi_1^2 + \xi_2^2) = (1-2a)(\xi_1^2 + \xi_2^2),
\end{aligned}$$

indeed $\xi = \xi_1 + \xi_2$ such that $|\xi| \leq |\xi_1| + |\xi_2| \leq 2\max\{\xi_1, \xi_2\}$. Hence,

$$3\max\{|\lambda|, |\lambda_1|, |\lambda_2|\} \geq (1-2a)\max\{\xi_1^2, \xi_2^2\} \geq \frac{1-2a}{4}\xi^2.$$

Therefore, supposing that $|\xi| \geq 1$, we have

$$\frac{1}{\max\{|\lambda|, |\lambda_1|, |\lambda_2|\}} \leq \frac{c}{|\xi|^2}.$$

Now, we define the regions $\mathcal{S}_i$,

$$\mathcal{S}_1 = \Big\{|\xi| \geq 1, |\lambda| = \max\{|\lambda|, |\lambda_1|, |\lambda_2|\}\Big\} \cup \Big\{|\xi| \leq 1\Big\} \subset \mathbb{R}^4_{\xi,\tau,\xi_2,\tau_2}, \tag{30}$$

$$\mathcal{S}_2 = \Big\{|\xi| \geq 1, |\lambda_1| = \max\{|\lambda|, |\lambda_1|, |\lambda_2|\}\Big\} \subset \mathbb{R}^4_{\xi,\tau,\xi_2,\tau_2}, \tag{31}$$

$$\mathcal{S}_3 = \Big\{|\xi| \geq 1, |\tau_2 + \xi^2| = \max\{|\lambda|, |\lambda_1|, |\lambda_2|\}\Big\} \subset \mathbb{R}^4_{\xi,\tau,\xi_2,\tau_2}. \tag{32}$$

For $\kappa \geq 0$, we have $\langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \leq \langle \xi \rangle^{-2\kappa}$ and in this case

$$J_4 \leq \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-2\kappa+4d} \chi_{\mathcal{S}_1}}{\langle \tau + \xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2. \tag{33}$$

Therefore, J_4 is bounded since $s - \kappa + 2d < 0$ for $s - \kappa < 2d$.

Note that J_5 and J_6 satisfy,

$$J_5 \leq \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-2\kappa-4b} \chi_{\mathcal{S}_2}}{\langle \tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2 \rangle^{2d}} d\xi \tag{34}$$

and

$$J_6 \leq \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-2\kappa-4b} \chi_{\mathcal{S}_3}}{\langle \tau_1 + a\xi^2 + \xi_2^2 \rangle^{2d}} d\xi_2, \tag{35}$$

and that they are bounded since $s - \kappa < 2b$ and $2d > \frac{1}{2}$, that is, $b < \frac{3}{4}$.

When $\kappa < 0$ we analyse the following subcases:

1. Considering $|\xi_1| \leq \frac{2}{3}|\xi_2|$, we have $\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa} \leq \langle\xi_2\rangle^{-4\kappa}$. Moreover, $|\xi_2| \leq |\xi_1| + |\xi| \leq \frac{2|\xi_2|}{3} + |\xi|$, hence $|\xi_2| \leq 3|\xi|$. Therefore,
$$\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa} \leq \langle\xi\rangle^{2s-4\kappa}.$$
2. Supposing $|\xi_2| \leq \frac{2}{3}|\xi_1|$, we have
$$\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa} \leq \langle\xi\rangle^{2s-4\kappa}.$$
3. The last case, $\frac{2}{3}|\xi_2| < |\xi_1| < \frac{3}{2}|\xi_2|$.
   (a) If $\xi_1, \xi_2 \geq 0$, then $\frac{2}{3}\xi_2 < \xi_1 < \frac{3}{2}\xi_2 \Longrightarrow \frac{5}{3}\xi_2 < \xi < \frac{5}{2}\xi_2$. Hence,
$$\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa} \leq \langle\xi\rangle^{2s-4\kappa}.$$
   (b) If $\xi_1, \xi_2 \leq 0$, then $\frac{-2}{3}\xi_2 < -\xi_1 < \frac{-3}{2}\xi_2 \Longrightarrow \frac{-5}{3}\xi_2 < -\xi < \frac{-5}{2}\xi_2$, thus $|\xi_2| < \frac{3}{5}|\xi|$. Hence,
$$\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa} \leq \langle\xi\rangle^{2s-4\kappa}.$$
   (c) If $\xi_1 > 0$ and $\xi_2 < 0$, then $\frac{-2}{3}\xi_2 < \xi_1 < \frac{-3}{2}\xi_2 \Longrightarrow \frac{1}{3}\xi_2 < \xi < \frac{-1}{2}\xi_2 \Longrightarrow |\xi| < \frac{1}{2}|\xi_2|$.
   (d) If $\xi_1 < 0$ and $\xi_2 > 0$, then $\frac{2}{3}\xi_2 < -\xi_1 < \frac{3}{2}\xi_2 \Longrightarrow \frac{-1}{3}\xi_2 < -\xi < \frac{1}{2}\xi_2$, consequently $|\xi| < \frac{1}{2}|\xi_2|$.

The cases (1), (2), [3(a)], and [3(b)] are true for $\kappa < 0$ and $s < 2\kappa + 1$.

Indeed, given $\mathcal{A} \subset \mathbb{R}^4$ the set of the elements of $\mathbb{R}^4$ that satisfies one of conditions (1), (2), [3(a)], or [3(b)], given $\mathcal{B} = \mathbb{R}^4 \setminus \mathcal{A}$. Now consider $\mathcal{A}_i = \mathcal{S}_i \cap \mathcal{A}$ and $\mathcal{B}_i = \mathcal{S}_i \cap \mathcal{B}$.

Analyzing the restrictions $\mathcal{A}_i$, we get
$$\begin{aligned} J_4 &= \frac{1}{\langle\lambda\rangle^{2d}} \int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa}\chi_{\mathcal{B}_1}}{\langle\tau + \xi_2^2 - 2\xi\xi_2 + \xi^2\rangle^{2b}} d\xi_2 \\ &\leq \int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s-4\kappa-4d}\chi_{\mathcal{A}_1}}{\langle\tau + \xi_2^2 - 2\xi\xi_2 + \xi^2\rangle^{2b}} d\xi_2. \end{aligned}$$

Then J_4 is bounded for $s \leq 2\kappa + 2d$ and $b < 3/4$.
$$\begin{aligned} J_5 &= \frac{1}{\langle\lambda_2\rangle^{2b}} \int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa}\chi_{\mathcal{A}_2}}{\langle\tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2\rangle^{2d}} d\xi \\ &\leq \int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s-4\kappa-4b}\chi_{\mathcal{A}_2}}{\langle\tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2\rangle^{2d}} d\xi \end{aligned}$$
and J_5 is bounded for $s \leq 2\kappa + 2b$ and $1/2 < b$.
$$\begin{aligned} J_6 &= \frac{1}{\langle\lambda_1\rangle^{2b}} \int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa}\chi_{\mathcal{A}_3}}{\langle\tau_1 + a\xi^2 + \xi_2^2\rangle^{2d}} d\xi_2 \\ &\leq \int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s-4\kappa-4b}\chi_{\mathcal{A}_3}}{\langle\tau_1 + a\xi^2 + \xi_2^2\rangle^{2d}} d\xi_2. \end{aligned}$$

Then J_6 is also bounded for $s \leq 2\kappa + 2b$ and $1/2 < b$.

To analyze the remaining cases (which is equivalent to supposing $|\xi| < \frac{1}{2}|\xi_2|$ and $|\xi_1| \sim |\xi_2|$) let us consider them as regions $\mathcal{B}_i$

We start by estimating J_4,
$$\begin{aligned} J_4 &= \frac{1}{\langle\lambda\rangle^{2d}} \int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa}\chi_{\mathcal{B}_1}}{\langle\tau + \xi_2^2 - 2\xi\xi_2 + \xi^2\rangle^{2b}} d\xi_2 \\ &\leq \int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s-4d}\langle\xi_1\rangle^{-4\kappa}\chi_{\mathcal{B}_1}}{\langle\tau + \xi_2^2 - 2\xi\xi_2 + \xi^2\rangle^{2b}} d\xi_2 \\ &\leq \int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s-4d}\langle\xi_1\rangle^{-4\kappa}\chi_{\mathcal{B}_1}}{2|\xi_2 - \xi|\langle\eta\rangle^{2b}} d\eta. \end{aligned}$$

Now, $|\xi_2 - \xi| \geq |\xi_2| - |\xi| \geq \frac{1}{2}|\xi_2| \sim \frac{1}{2}|\xi_1|$.

Hence, $J_4 \leq \langle\xi\rangle^{2s-4d}\langle\xi_1\rangle^{-4\kappa-1}\int_{\mathbb{R}} \frac{d\eta}{\langle\eta\rangle^{2b}}$, that is bounded because $2b > 1$ and $2s \leq 4\kappa + 2$.

$$\langle\xi\rangle^{2s-4d}\langle\xi_1\rangle^{-4\kappa-1} \leq \langle\xi\rangle^{2s-4\kappa-4d-1}.$$

We continue to estimate J_5,

$$\begin{aligned} J_5 &= \frac{1}{\langle\lambda_2\rangle^{2b}}\int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa}\chi_{\mathcal{B}_2}}{\langle\tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2\rangle^{2d}}d\xi \\ &\leq \frac{1}{\langle\lambda_2\rangle^{2b}}\int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s}\langle\xi_2\rangle^{-4\kappa}\chi_{\mathcal{B}_2}}{\langle\tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2\rangle^{2d}}d\xi. \end{aligned}$$

Setting $\eta = \tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2$ such that $d\eta = 2(\xi_2 + (a-1)\xi)d\xi$. Now, as $0 < a < \frac{1}{2}$, it follows $|a-1| < 1$ and therefore $|\xi_2 + (a-1)\xi| \geq \frac{1}{2}|\xi_2|$. Observe still that

$$\begin{aligned} |\eta| &= |\tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2| \\ &= |\lambda_2 + ((a-1)\xi^2 - 2\xi_2^2 + 2\xi\xi_2)| \\ &\leq |\lambda_2| + |(a-1)\xi^2 - 2\xi_2^2 + 2\xi\xi_2| \leq |\tau_2 + \xi_2| + 4|\xi_2|^2 \\ &\leq c|\lambda_2|. \end{aligned}$$

Thus,

$$\begin{aligned} J_5 &\leq \frac{1}{\langle\lambda_2\rangle^{2b}}\int_{\langle\eta\rangle \leq c\langle\lambda_2\rangle} \frac{\langle\xi\rangle^{2s}\langle\xi_2\rangle^{-4\kappa-1}}{\langle\eta\rangle^{2d}}d\eta \\ &\leq \frac{1}{\langle\lambda_2\rangle^{2b}}\int_{\langle\eta\rangle \leq c\langle\lambda_2\rangle} \frac{\langle\xi_2\rangle^{\max\{0,2s\}-4\kappa-1}}{\langle\eta\rangle^{2d}}d\eta, \quad \text{because } |\xi| < \frac{1}{2}|\xi_2| \\ &\leq \langle\xi_2\rangle^{\max\{0,2s\}-4\kappa-1}\frac{\langle\lambda_2\rangle^{2d}}{\langle\lambda_2\rangle^{2b}} \\ &\leq \langle\xi_2\rangle^{\max\{0,2s\}-4\kappa-1}\langle\lambda_2\rangle^{-2b+2d} \leq \langle\xi_2\rangle^{\max\{0,2s\}-4\kappa-1-2b+2d}. \end{aligned}$$

We prove J_6. Remember that

$$\begin{aligned} J_6 &= \frac{1}{\langle\lambda_1\rangle^{2b}}\int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa}\chi_{\mathcal{B}_3}}{\langle\tau_1 + a\xi^2 + \xi_2^2\rangle^{2d}}d\xi_2 \\ &\leq \frac{1}{\langle\lambda_1\rangle^{2b}}\int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-4\kappa}\chi_{\mathcal{B}_3}}{\langle\tau_1 + a\xi^2 + \xi_2^2\rangle^{2d}}d\xi_2. \end{aligned}$$

Let $\eta = \tau_1 + a\xi^2 + \xi_2^2$ such that $d\eta = 2\xi_2 d\xi_2$. Now,

$$\begin{aligned} |\eta| &= |\tau_1 + a\xi^2 + \xi_2^2| \\ &= |(\lambda_1) + (a\xi^2 + \xi_2^2 - \xi_1^2)| \\ &\leq c|\lambda_1|. \end{aligned}$$

And using that $|\xi_1| \sim |\xi_2|$, we obtain

$$\begin{aligned} J_6 &\leq \frac{1}{\langle\lambda_1\rangle^{2b}}\int_{\langle\eta\rangle \leq c\langle\lambda_1\rangle} \frac{\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-4\kappa}}{|\xi_1|\langle\eta\rangle^{2d}}d\xi_2 \\ &\leq \langle\xi_1\rangle^{\max\{0,2s\}-4\kappa-1}\frac{\langle\lambda_1\rangle^{2d}}{\langle\lambda_1\rangle^{2b}} \\ &\leq \langle\xi_1\rangle^{\max\{0,2s\}-4\kappa-1-2b+2d}. \end{aligned}$$

As $1/2 < b < 3/4$ and $1/4 < d < 1/2$, we get $-1 < -2b + 2d < 0$ and hence we can take b and d so that $2s - 4\kappa - 1 - 2b + 2d < 0$ if $s < 2k + 1$.

Then, we completed the proof of Proposition 2. □

Remark 2. The lines $s = -\kappa - 1$ *and* $s = 2\kappa + 1$ *intersect each other at the point where* $\kappa = -\frac{2}{3}$.

B. Bilinear estimates for $\sigma < 2$

For $a > 1/2$, we have some results present below.

Proposition 3. Assume that $a > 1/2$ (equivalently $\sigma < 2$), $u \in X^{\kappa,b}$, and $v \in X_a^{s,b}$, then the bilinear estimate below holds if $1/2 < b < 3/4$, $1/4 < d < 1/2$, and $|\kappa| - s \leq 1/2$,

$$\|\overline{u} \cdot v\|_{X^{\kappa,-d}} \leq C\|u\|_{X^{\kappa,b}} \cdot \|v\|_{X^{s,b}}. \tag{36}$$

The second estimate tells us that

Proposition 4. Let $a > 1/2$ (equivalently $\sigma < 2$), u, $\tilde{u} \in X^{\kappa,b}$ with $1/2 < b < 3/4$ and $1/4 < d < 1/2$. The estimate

$$\|u \cdot \tilde{u}\|_{X^{s,-d}} \leq C\|u\|_{X^{\kappa,b}} \cdot \|\tilde{u}\|_{X^{\kappa,b}} \tag{37}$$

holds for $s \leq \min\{\kappa + 1/2, \ 2\kappa + 1/2\}$.

Proof of Proposition 3: We start by considering the dispersion relation.

Note that

$$\begin{aligned}|\omega - \omega_1 - \omega_2| &= |\xi^2 + \xi_1^2 - a\xi_2^2| \\ &\geq |2\xi^2 - 2\xi\xi_2 + (1-a)\xi_2^2|, \quad \text{using } a > \frac{1}{2}, \text{ we have} \\ &= 2|\xi - \mu_a\xi_2| \cdot |\xi - (1-\mu_a)\xi_2|, \text{ where } \mu_a = \frac{1 - \sqrt{2a-1}}{2}.\end{aligned}$$

Note that the above dispersion relation has two regions: the lines $\xi = \mu_a\xi_2$ and $\xi = (1-\mu_a)\xi_2$ making it difficult to use the relationship. Observe that if $a = \frac{1}{2}$, then $\mu_a = 1 - \mu_a = \frac{1}{2}$ and if $a = 1$, then $\mu_a = 0$ (the case $a = \frac{1}{2}$ will be treated separately, while the case $a = 1$ does not require much attention despite being the case without modification).[1]

Before doing it, consider

$$\begin{aligned}\mathcal{A}_1 &= \{|\xi_2| \leq 1\} \subset \mathbb{R}^4, \\ \mathcal{A}_2 &= \left\{|\xi_2| \geq 1, |(1-a)\xi_2 - \xi| > \frac{2a-1}{4}|\xi_2|\right\} \subset \mathbb{R}^4, \\ \mathcal{A}_3 &= \left\{|\xi_2| \geq 1, \left|\xi - \frac{1}{2}\xi_2\right| > \frac{2a-1}{4}|\xi_2|\right\} \subset \mathbb{R}^4.\end{aligned}$$

Note that if $\left|\xi - \frac{1}{2}\xi_2\right| \leq \frac{2a-1}{4}|\xi_2|$, $\left|\xi - \frac{1}{2}\xi_2\right| \leq \frac{2a-1}{4}|\xi_2|$, and $|\xi_2| \geq 1$, then

$$\begin{aligned}\left(a - \frac{1}{2}\right)|\xi_2| &= \left|\left(\xi - \frac{1}{2}\xi_2\right) + ((1-a)\xi_2 - \xi)\right| \\ &\leq \frac{2a-1}{4}|\xi_2| + \frac{2a-1}{4}|\xi_2| = \frac{1}{2}\left(a - \frac{1}{2}\right)|\xi_2|.\end{aligned}$$

This contradiction implies $\mathbb{R}^4 = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$.

Now consider,

$$\begin{aligned}\mathcal{A}_{3,1} &= \mathcal{A}_3 \cap \{|\omega| \geq \max\{|\omega_1|, |\omega_2|\}\}, \\ \mathcal{A}_{3,2} &= \mathcal{A}_3 \cap \{|\omega_2| \geq \max\{|\omega_1|, |\omega|\}\}, \\ \mathcal{A}_{3,3} &= \mathcal{A}_3 \cap \{|\omega_1| \geq \max\{|\omega|, |\omega_2|\}\}.\end{aligned}$$

Remember that $|2\xi^2 + \xi_2^2 - 2\xi\xi_2| \leq 3\max\{|\omega|, |\omega_1|, |\omega_2|\}$.

Now, we define the regions $\mathcal{R}_i$ (analogous to the proof of proposition 1). Let $\mathcal{R}_1 = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_{3,1}$, $\mathcal{R}_2 = \mathcal{A}_{3,2}$ and $\mathcal{R}_3 = \mathcal{A}_{3,3}$.

We will show that J_1 is bounded. Indeed, if $|\xi_2| \leq 1$, then J_1 is equivalent to

$$\frac{1}{\langle\omega\rangle^{2d}} \int_{|\xi_2| \leq 1} \frac{1}{\langle\tau - (a-1)\xi_2^2 - 2\xi\xi_2 + \xi^2\rangle^{2b}} d\xi_2 \leq c.$$

If $|\xi_2| \geq 1$, then

$$J_1 \leq \frac{1}{\langle\omega\rangle^{2d}} \int_{|\xi_2|\geq 1} \frac{\langle\xi_2\rangle^{-2s+2|\kappa|}\chi_{\mathcal{A}_2}}{\langle\tau-(a-1)\xi_2^2-2\xi\xi_2+\xi^2\rangle^{2b}} d\xi_2.$$

Changing the variable $\eta = \tau - (a-1)\xi_2^2 - 2\xi\xi_2 + \xi^2$, we get

$$d\eta = -2((1-a)\xi_2 - \xi)d\xi_2$$

and, using the fact that $|\kappa| - s \leq 1/2$, obtain the following equations:

$$\begin{aligned}\frac{1}{\langle\omega\rangle^{2d}} \int_{|\xi_2|\geq 1} \frac{\langle\xi_2\rangle^{-2s+2|\kappa|}\chi_{\mathcal{A}_2}}{\langle\tau-(a-1)\xi_2^2-2\xi\xi_2+\xi^2\rangle^{2b}} d\xi_2 &\leq c\frac{1}{\langle\omega\rangle^{2d}} \int_{|\xi_2|\geq 1} \frac{\langle\xi_2\rangle^{-2s+2|\kappa|-1}\chi_{\mathcal{A}_2}}{\langle\eta\rangle^{2b}} d\eta \\ &\leq c\frac{1}{\langle\omega\rangle^{2d}} \int_{\mathbb{R}} \frac{1}{\langle\eta\rangle^{2b}} d\eta \leq c.\end{aligned}$$

Now, note that in $\mathcal{A}_{3,1}$ we have

$$|(1-a)\xi_2 - \xi| = \left|\frac{1}{2}\xi_2 - \xi + \left(\frac{1}{2}-a\right)\xi_2\right| \geq \left(a-\frac{1}{2}\right)|\xi_2| - \frac{2a-1}{4}|\xi_2| \geq c|\xi_2|.$$

To complete the estimate of J_1, we change variable to get

$$\begin{aligned}\frac{1}{\langle\omega\rangle^{2d}} \int_{|\xi_2|\geq 1} \frac{\langle\xi_2\rangle^{-2s+2|\kappa|}\chi_{\mathcal{A}_{3,1}}}{\langle\tau-(a-1)\xi_2^2-2\xi\xi_2+\xi^2\rangle^{2b}} d\xi_2 &\leq c\frac{1}{\langle\omega\rangle^{2d}} \int_{|\xi_2|\geq 1} \frac{\langle\xi_2\rangle^{-2s+2|\kappa|-1}\chi_{\mathcal{A}_{3,1}}}{\langle\eta\rangle^{2b}} d\eta \\ &\leq c\frac{1}{\langle\omega\rangle^{2d}} \int_{\mathbb{R}} \frac{1}{\langle\eta\rangle^{2b}} d\eta \leq c.\end{aligned}$$

To prove that J_2 is bounded just observe that

$$\begin{aligned}\frac{1}{\langle\tau+a\xi^2\rangle^{2b}} \int_{\mathbb{R}} \frac{\langle\xi_2\rangle^{-2s+2|\kappa|}\chi_{\mathcal{R}_2}}{\langle\tau_2+2\xi^2+\xi_2^2-2\xi\xi_2\rangle^{2d}} d\xi &= \frac{1}{\langle\omega_2\rangle^{2b}} \int_{\mathbb{R}} \frac{\langle\xi_2\rangle^{-2s+2|\kappa|}\chi_{\mathcal{A}_{3,2}}}{\langle\tau_2+2\xi^2+\xi_2^2-2\xi\xi_2\rangle^{2d}} d\xi \\ &\leq \frac{1}{\langle\omega_2\rangle^{2b}} \int_{\langle\eta\rangle\leq 4\langle\omega_2\rangle} \frac{\langle\xi_2\rangle^{-2s+2|\kappa|-1}}{\langle\eta\rangle^{2d}} d\eta \\ &\leq \frac{1}{\langle\omega_2\rangle^{2b-2d}}.\end{aligned}$$

In the first inequality above, we made the change of variable $\eta = \tau_2 + 2\xi^2 + \xi_2^2 - 2\xi\xi_2$ and used the fact that

$$|\eta| = |\omega_2 + (\omega - \omega_1 - \omega_2)| \leq 4|\omega_2|.$$

We estimate J_3. Analogous to the last estimate, we get

$$\begin{aligned}\frac{1}{\langle\tau-\xi_1^2\rangle^{2b}} \int_{\mathbb{R}} \frac{\langle\xi_2\rangle^{-2s+2|\kappa|}\chi_{\mathcal{R}_3}}{\langle\tau_1-a\xi_2^2+\xi^2\rangle^{2d}} d\xi_2 &= \frac{1}{\langle\omega_1\rangle^{2b}} \int_{|\xi_2|>1} \frac{\langle\xi_2\rangle^{-2s+2|\kappa|}\chi_{\mathcal{A}_{3,3}}}{\langle\tau_1-a\xi_2^2+\xi^2\rangle^{2d}} d\xi_2 \\ &\leq \frac{1}{\langle\omega_1\rangle^{2b}} \int_{\langle\eta\rangle\leq 4\langle\omega_1\rangle} \frac{\langle\xi_2\rangle^{-2s+2|\kappa|-1}}{\langle\eta\rangle^{2d}} d\eta \\ &\leq \frac{1}{\langle\omega_1\rangle^{2b-2d}}.\end{aligned}$$

Note that we used the fact that $\tau_1 - a\xi_2^2 + \xi^2 = \omega_1 + (\omega - \omega_1 - \omega_2)$.

This finishes the proof of the first inequality. □

Proof of Proposition 4. Initially, we have that

$$\begin{aligned}|\lambda - \lambda_1 - \lambda_2| &= |a\xi^2 - \xi_1^2 - \xi_2^2| \\ &\geq |2\xi_2^2 - 2\xi\xi_2 + (1-a)\xi^2| \quad \text{using } a > \frac{1}{2} \text{ we have} \\ &= 2|\xi_2 - \mu_a\xi| \cdot |\xi_2 - (1-\mu_a)\xi|, \text{ where } \mu_a = \frac{1-\sqrt{2a-1}}{2}.\end{aligned}$$

The dispersion relation above is zero in two straight lines.

Now, we define

$$\mathcal{B}_1 = \{|\xi| \le 1\} \subset \mathbb{R}^4,$$
$$\mathcal{B}_2 = \left\{|\xi| \ge 1, \left|\xi_2 - \frac{1}{2}\xi\right| > \frac{2a-1}{4}|\xi|\right\} \subset \mathbb{R}^4,$$
$$\mathcal{B}_3 = \left\{|\xi| \ge 1, |(1-a)\xi - \xi_2| > \frac{2a-1}{4}|\xi|\right\} \subset \mathbb{R}^4.$$

Note that if $\left|\xi_2 - \frac{1}{2}\xi\right| \le \frac{2a-1}{4}|\xi|$ and $\left|\xi_2 - \frac{1}{2}\xi\right| \le \frac{2a-1}{4}|\xi|$ and still $|\xi| > 1$, then

$$\begin{aligned}\left(a - \frac{1}{2}\right)|\xi| &= \left|\left(\xi_2 - \frac{1}{2}\xi\right) + ((1-a)\xi - \xi_2)\right| \\ &\le \frac{2a-1}{4}|\xi| + \frac{2a-1}{4}|\xi| = \frac{1}{2}\left(a - \frac{1}{2}\right)|\xi|.\end{aligned}$$

Again, this contradiction implies $\mathbb{R}^4 = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$.

Now, consider

$$\mathcal{B}_{3,1} = \mathcal{B}_3 \cap \{|\lambda| \ge \max\{|\lambda_1|, |\lambda_2|\}\},$$
$$\mathcal{B}_{3,2} = \mathcal{B}_3 \cap \{|\lambda_2| \ge \max\{|\lambda_1|, |\lambda|\}\},$$
$$\mathcal{B}_{3,3} = \mathcal{B}_3 \cap \{|\lambda_1| \ge \max\{|\lambda|, |\lambda_2|\}\}.$$

We define the regions $\mathcal{S}_i$ (analogous to the proof of proposition 2), setting $\mathcal{S}_1 = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_{3,1}$, $\mathcal{S}_2 = \mathcal{B}_{3,2}$ and $\mathcal{S}_3 = \mathcal{B}_{3,3}$.

For $\kappa \ge 0$, we have $\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa} \le \langle\xi\rangle^{-2\kappa}$

$$J_4 \le \frac{1}{\langle\lambda\rangle^{2d}} \int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s-2\kappa}\chi_{\mathcal{S}_1}}{\langle\tau + \xi_2^2 - 2\xi\xi_2 + \xi^2\rangle^{2b}} d\xi_2, \tag{38}$$

$$J_5 \le \frac{1}{\langle\lambda_2\rangle^{2b}} \int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s-2\kappa}\chi_{\mathcal{S}_2}}{\langle\tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2\rangle^{2d}} d\xi, \tag{39}$$

$$J_6 \le \frac{1}{\langle\lambda_1\rangle^{2b}} \int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s-2\kappa}\chi_{\mathcal{S}_3}}{\langle\tau_1 + a\xi^2 + \xi_2^2\rangle^{2d}} d\xi_2. \tag{40}$$

To complete the proof that J_4 is bounded it is sufficient to show that (38) satisfies:

$$\frac{1}{\langle\lambda\rangle^{2d}} \int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s-2\kappa}\chi_{\mathcal{B}_1}}{\langle\tau + \xi_2^2 - 2\xi\xi_2 + \xi^2\rangle^{2b}} d\xi_2 \le \frac{1}{\langle\lambda\rangle^{2d}} \int_{\mathbb{R}} \frac{1}{\langle\tau + \xi_2^2 - 2\xi\xi_2 + \xi^2\rangle^{2b}} d\xi_2 \le c,$$

$$\frac{1}{\langle\lambda\rangle^{2d}} \int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s-2\kappa}\chi_{\mathcal{B}_2}}{\langle\tau + \xi_2^2 - 2\xi\xi_2 + \xi^2\rangle^{2b}} d\xi_2 \le \frac{1}{\langle\lambda\rangle^{2d}} \int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s-2\kappa-1}}{\langle\eta\rangle^{2b}} d\xi_2 \le c, \text{ and}$$

$$\frac{1}{\langle\lambda\rangle^{2d}} \int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s-2\kappa}\chi_{\mathcal{B}_{3,1}}}{\langle\tau + \xi_2^2 - 2\xi\xi_2 + \xi^2\rangle^{2b}} d\xi_2 \le \frac{1}{\langle\lambda\rangle^{2d}} \int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s-2\kappa-1}}{\langle\eta\rangle^{2b}} d\xi_2 \le c.$$

In the estimates above, we used the fact $b > 1/2$ and also the fact that

$$\begin{aligned}\left|\xi_2 - \frac{1}{2}\xi\right| &= \left|(1-a)\xi - \xi_2 + \left(a - \frac{1}{2}\right)\xi\right| \\ &\ge \left(a - \frac{1}{2}\right)|\xi| - |(1-a)\xi - \xi_2| \\ &\ge \left(a - \frac{1}{2}\right)|\xi| - \frac{1}{2}\left(a - \frac{1}{2}\right)|\xi| = \frac{2a-1}{4}|\xi|.\end{aligned}$$

Let us estimate (39), using the fact that

$$\eta = \tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2 = \lambda_2 + (\lambda - \lambda_1 - \lambda_2),$$

which give us $d\eta = 2((1-a)\xi - \xi_2)d\xi$, so

$$\begin{aligned}\frac{1}{\langle\lambda_2\rangle^{2b}}\int_{\mathbb{R}}\frac{\langle\xi\rangle^{2s-2\kappa}\chi_{\mathcal{B}_{3,2}}}{\langle\tau_2+(a-1)\xi^2-\xi_2^2+2\xi\xi_2\rangle^{2d}}d\xi &\le \frac{1}{\langle\lambda_2\rangle^{2b}}\int_{\langle\eta\rangle\le 4\langle_2\lambda\rangle}\frac{\langle\xi\rangle^{2s-2\kappa-1}}{\langle\eta\rangle^{2d}}d\eta\\ &\le \frac{1}{\langle\lambda_2\rangle^{2b-2d}}\le c.\end{aligned}$$

Now let us estimate (40). This is completely analogous to the previous estimate.

$$\begin{aligned}\frac{1}{\langle\lambda_1\rangle^{2b}}\int_{\mathbb{R}}\frac{\langle\xi\rangle^{2s-2\kappa}\chi_{\mathcal{S}_3}}{\langle\tau_1+a\xi^2+\xi_2^2\rangle^{2d}}d\xi_2 &\le \frac{1}{\langle\lambda_1\rangle^{2b}}\int_{\langle\eta\rangle\le 4\langle\lambda_1\rangle}\frac{\langle\xi\rangle^{2s-2\kappa-1}}{\langle\eta\rangle^{2d}}d\eta\\ &\le \frac{1}{\langle\lambda_1\rangle^{2b-2d}}\le c.\end{aligned}$$

This concludes the case $\kappa \ge 0$.

The case $\kappa < 0$ will be separated into sub-cases:

1. Supposing $|\xi_1| \le \frac{2}{3}|\xi_2|$, then, $\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa} \le \langle\xi_2\rangle^{-4\kappa}$. Moreover,$|\xi_2| \le |\xi_1| + |\xi| \le \frac{2|\xi_2|}{3} + |\xi|$, hence $|\xi_2| \le 3|\xi|$. Therefore,

$$\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa} \le \langle\xi\rangle^{2s-4\kappa}.$$

2. Supposing $|\xi_2| \le \frac{2}{3}|\xi_1|$, we have the same result, that is,

$$\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa} \le \langle\xi\rangle^{2s-4\kappa}.$$

3. For the case, $\frac{2}{3}|\xi_2| < |\xi_1| < \frac{3}{2}|\xi_2|$, we need to do the following:
   (a) If $\xi_1, \xi_2 \ge 0$, then $\frac{2}{3}\xi_2 < \xi_1 < \frac{3}{2}\xi_2 \Longrightarrow \frac{5}{3}\xi_2 < \xi < \frac{5}{2}\xi_2$. Hence,

$$\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa} \le \langle\xi\rangle^{2s-4\kappa}.$$

   (b) If $\xi_1, \xi_2 \le 0$, then $\frac{-2}{3}\xi_2 < -\xi_1 < \frac{-3}{2}\xi_2 \Longrightarrow \frac{-5}{3}\xi_2 < -\xi < \frac{-5}{2}\xi_2$, so $|\xi_2| < \frac{3}{5}|\xi|$. Hence,

$$\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa} \le \langle\xi\rangle^{2s-4\kappa}.$$

   (c) If $\xi_1 > 0$ and $\xi_2 < 0$, then $\frac{-2}{3}\xi_2 < \xi_1 < \frac{-3}{2}\xi_2 \Longrightarrow \frac{1}{3}\xi_2 < \xi < \frac{-1}{2}\xi_2$, now $|\xi| < \frac{1}{2}|\xi_2|$.
   (d) If $\xi_1 < 0$ and $\xi_2 > 0$, then $\frac{2}{3}\xi_2 < -\xi_1 < \frac{3}{2}\xi_2 \Longrightarrow \frac{-1}{3}\xi_2 < -\xi < \frac{1}{2}\xi_2$, which give us $|\xi| < \frac{1}{2}|\xi_2|$.

The cases (1), (2), [3(a)], and [3(b)] are valid for $\kappa < 0$ and $s < 2\kappa + \frac{1}{2}$.

Indeed, let $\mathcal{C} \subset \mathbb{R}^4$ be the set of element $\mathbb{R}^4$ that satisfies one of the conditions (1), (2), [3(a)], or [3(b)]. Now consider $\mathcal{C}_i = \mathcal{S}_i \cap \mathcal{C}$.

Analyzing the restrictions on $\mathcal{C}_i$, we get

$$\begin{aligned}\frac{1}{\langle\lambda\rangle^{2d}}\int_{\mathbb{R}}\frac{\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa}\chi_{\mathcal{C}_1}}{\langle\tau+\xi_2^2-2\xi\xi_2+\xi^2\rangle^{2b}}d\xi_2 &\le \frac{1}{\langle\lambda\rangle^{2d}}\int_{\mathbb{R}}\frac{\langle\xi\rangle^{2s-4\kappa}\chi_{\mathcal{C}_1}}{\langle\tau+\xi_2^2-2\xi\xi_2+\xi^2\rangle^{2b}}d\xi_2\\ &\le \frac{1}{\langle\lambda\rangle^{2d}}\int_{\mathbb{R}}\frac{\langle\xi\rangle^{2s-4\kappa-1}}{\langle\eta\rangle^{2b}}d\xi_2\\ &\le c, \quad \text{because } 1/2 < b < 1 \text{ and } s < 2\kappa + 1/2.\end{aligned}$$

$$\begin{aligned}\frac{1}{\langle\lambda_2\rangle^{2b}}\int_{\mathbb{R}}\frac{\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa}\chi_{\mathcal{C}_2}}{\langle\tau_2+(a-1)\xi^2-\xi_2^2+2\xi\xi_2\rangle^{2d}}d\xi &\le \frac{1}{\langle\lambda_2\rangle^{2b}}\int_{\langle\eta\rangle\le 4\langle\lambda_2\rangle}\frac{\langle\xi\rangle^{2s-4\kappa-1}}{\langle\eta\rangle^{2d}}d\eta\\ &\le \frac{1}{\langle\lambda_2\rangle^{2b-2d}}\le c.\end{aligned}$$

$$\frac{1}{\langle\lambda_1\rangle^{2b}}\int_{\mathbb{R}}\frac{\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa}\chi_{\mathcal{C}_3}}{\langle\tau_1+a\xi^2+\xi_2^2\rangle^{2d}}d\xi_2 \le \frac{1}{\langle\lambda_1\rangle^{2b}}\int_{\langle\eta\rangle\le 4\langle\lambda_1\rangle}\frac{\langle\xi\rangle^{2s-4\kappa-1}}{\langle\eta\rangle^{2d}}d\eta \\ \le \frac{1}{\langle\lambda_1\rangle^{2b-2d}} \le c.$$

Consider $\mathcal{D}=\mathbb{R}^4\setminus\mathcal{C}$ and $\mathcal{D}_i=\mathcal{S}_i\cap\mathcal{D}$. To obtain the other cases (which is equivalent to supposing $|\xi|<\frac{1}{2}|\xi_2|$ and $|\xi_1|\sim|\xi_2|$) let us consider the regions $\mathcal{D}_i$.

We begin by estimating J_4.

$$\frac{1}{\langle\lambda\rangle^{2d}}\int_{\mathbb{R}}\frac{\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa}\chi_{\mathcal{D}_1}}{\langle\tau+\xi_2^2-2\xi\xi_2+\xi^2\rangle^{2b}}d\xi_2 \le \frac{1}{\langle\lambda\rangle^{2d}}\int_{\mathbb{R}}\frac{\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-4\kappa}\chi_{\mathcal{D}_1}}{\langle\tau+\xi_2^2-2\xi\xi_2+\xi^2\rangle^{2b}}d\xi_2 \\ \le \frac{1}{\langle\lambda\rangle^{2d}}\int_{\mathbb{R}}\frac{\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}}{\langle\eta\rangle^{2b}}d\eta.$$

Now, $|\xi_2-\xi|\ge|\xi_2|-|\xi|\ge\frac{1}{2}|\xi_2|\sim\frac{1}{2}|\xi_1|$.

Hence, $J_4\le\langle\xi\rangle^{2s-4d}\langle\xi_1\rangle^{-4\kappa-1}\int_{\mathbb{R}}\frac{d\eta}{\langle\eta\rangle^{2b}}$, the right-hand side is bounded because $2b>1$, $2s\le 4\kappa+2$, and $1/4<d<1/2$ in addition,

$$\langle\xi\rangle^{2s-4d}\langle\xi_1\rangle^{-4\kappa-1}\le\langle\xi\rangle^{2s-4\kappa-1-4d}\le\langle\xi\rangle^{1-4d}.$$

Estimating J_5:

$$J_5=\frac{1}{\langle\lambda_2\rangle^{2b}}\int_{\mathbb{R}}\frac{\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa}\chi_{\mathcal{B}_2}}{\langle\tau_2+(a-1)\xi^2-\xi_2^2+2\xi\xi_2\rangle^{2d}}d\xi \\ \le \frac{1}{\langle\lambda_2\rangle^{2b}}\int_{\mathbb{R}}\frac{\langle\xi\rangle^{2s}\langle\xi_2\rangle^{-4\kappa}\chi_{\mathcal{B}_2}}{\langle\tau_2+(a-1)\xi^2-\xi_2^2+2\xi\xi_2\rangle^{2d}}d\xi.$$

Setting $\eta=\tau_2+(a-1)\xi^2-\xi_2^2+2\xi\xi_2$, which give $d\eta=2(\xi_2+(a-1)\xi)d\xi$. As $0<a<\frac{1}{2}$, we have $|a-1|\le 1$ and therefore $|\xi_2+(a-1)\xi|\ge\frac{1}{2}|\xi_2|$. Also we note that

$$\begin{aligned}|\eta| &= |\tau_2+(a-1)\xi^2-\xi_2^2+2\xi\xi_2| \\ &= |(\lambda_2)+((a-1)\xi^2-2\xi_2^2+2\xi\xi_2)| \\ &\le |\lambda_2|+|(a-1)\xi^2-2\xi_2^2+2\xi\xi_2|\le|\tau_2+\xi_2|+4|\xi_2|^2 \\ &\le c|\lambda_2|.\end{aligned}$$

Hence,

$$\begin{aligned}J_5 &\le \frac{1}{\langle\lambda_2\rangle^{2b}}\int_{\langle\eta\rangle\le c\langle\lambda_2\rangle}\frac{\langle\xi\rangle^{2s}\langle\xi_2\rangle^{-4\kappa-1}}{\langle\eta\rangle^{2d}}d\eta \\ &\le \frac{1}{\langle\lambda_2\rangle^{2b}}\int_{\langle\eta\rangle\le c\langle\lambda_2\rangle}\frac{\langle\xi_2\rangle^{\max\{0,2s\}-4\kappa-1}}{\langle\eta\rangle^{2d}}d\eta \quad \text{because } |\xi|<\frac{1}{2}|\xi_2| \\ &\le \langle\xi_2\rangle^{\max\{0,2s\}-4\kappa-1}\frac{\langle\lambda_2\rangle^{1-2d}}{\langle\lambda_2\rangle^{2b}} \\ &\le \langle\xi_2\rangle^{\max\{0,2s\}-4\kappa-1}\langle\lambda_2\rangle^{1-2d-2b}\le\langle\xi_2\rangle^{\max\{0,2s\}-4\kappa-2}.\end{aligned}$$

Since $1-2b-2d<-1/2$.

Now, we estimate J_6. Remembering that

$$J_6=\frac{1}{\langle\lambda_1\rangle^{2b}}\int_{\mathbb{R}}\frac{\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa}\chi_{\mathcal{B}_3}}{\langle\tau_1+a\xi^2+\xi_2^2\rangle^{2d}}d\xi_2 \\ \le \frac{1}{\langle\lambda_1\rangle^{2b}}\int_{\mathbb{R}}\frac{\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-4\kappa}\chi_{\mathcal{B}_3}}{\langle\tau_1+a\xi^2+\xi_2^2\rangle^{2d}}d\xi_2.$$

Using $\eta = \tau_1 + a\xi^2 + \xi_2^2$, which give $d\eta = 2\xi_2 d\xi_2$. Now,

$$\begin{aligned} |\eta| &= |\tau_1 + a\xi^2 + \xi_2^2| \\ &= |(\lambda_1) + (a\xi^2 + \xi_2^2 - \xi_1^2)| \\ &\le c|\lambda_1|. \end{aligned}$$

By using the fact that $|\xi_1| \sim |\xi_2|$, we have

$$\begin{aligned} J_6 &\le \frac{1}{\langle\lambda_1\rangle^{2b}} \int_{\langle\eta\rangle \le c\langle\lambda_1\rangle} \frac{\langle\xi\rangle^{2s}\langle\xi_1\rangle^{-4\kappa}}{|\xi_1|\langle\eta\rangle^{2d}} d\xi_2 \\ &\le \langle\xi_1\rangle^{\max\{0,2s\}-4\kappa-1} \frac{\langle\lambda_1\rangle^{1-2d}}{\langle\lambda_1\rangle^{2b}} \\ &\le \langle\xi_1\rangle^{\max\{0,2s\}-4\kappa-2}. \end{aligned}$$

And this finishes the proof of Proposition 4. □

Remark 3. The lines $s = -\kappa - 1/2$ and $s = 2\kappa + 1/2$ intersect each other at the point $\kappa = -\frac{1}{3}$.

C. Bilinear estimates for $\sigma = 2$

Next we prove a new bilinear estimates for the interaction terms in the case $\sigma = 2$

Proposition 5. Assume that $a = 1/2$ (equivalently $\sigma = 2$). If $1/2 < b < 3/4$, $1/4 < d < 1/2$ and $|\kappa| \le s$, then for $u \in X^{\kappa,b}$ and $v \in X_a^{s,b}$, the estimate below

$$\|\overline{u} \cdot v\|_{X^{\kappa,-d}} \le C\|u\|_{X^{\kappa,b}} \cdot \|v\|_{X^{s,b}} \tag{41}$$

holds.

The second bilinear estimate tells us that

Proposition 6. Let $a = 1/2$ (equivalently $\sigma = 2$) and $u, \tilde{u} \in X^{\kappa,b}$, then

$$\|u \cdot \tilde{u}\|_{X^{s,-d}} \le C\|u\|_{X^{\kappa,b}} \cdot \|\tilde{u}\|_{X^{\kappa,b}} \tag{42}$$

holds if $1/2 < b < 3/4$, $1/4 < d < 1/2$ and $0 \le s \le \kappa$.

Proof of Proposition 5: We begin by noting that

$$\begin{aligned} |\omega - \omega_1 - \omega_2| &= \left|\xi^2 + \xi_1^2 - \frac{1}{2}\xi_2^2\right| \\ &= \left|2\xi^2 + 2\xi\xi_2 + \frac{1}{2}\xi_2^2\right| \\ &= 2\left|\xi + \frac{1}{2}\xi_2\right|^2. \end{aligned}$$

In this case, we do not have to take the dispersion relation. Then, we consider $\mathcal{R}_1 = \mathbb{R}^4$ and $\mathcal{R}_2 = \mathcal{R}_3 = \varnothing$. Thus, we only need to prove that J_1 is bounded. If $|\kappa| \le s$, then J_1 is equivalent to

$$\frac{1}{\langle\omega\rangle^{2d}} \int_{|\xi_2| \le 1} \frac{1}{\langle\tau - \frac{1}{2}\xi_2^2 - 2\xi\xi_2 + \xi^2\rangle^{2b}} d\xi_2 \le c,$$

since $b > 1/2$ and $d > 0$. This finishes the proof of the proposition. □

Proof of Proposition 6: As in the previous case, we cannot take advantage of the dispersion relation. So let us take $\mathcal{S}_1 = \mathbb{R}^4$ and $\mathcal{S}_2 = \mathcal{S}_3 = \varnothing$. Note that it is enough to estimate J_4. Initially assume that $\kappa \ge 0$, so we get $\langle\xi_1\rangle^{-2\kappa}\langle\xi_2\rangle^{-2\kappa} \le \langle\xi\rangle^{-2\kappa}$

$$J_4 \le \frac{1}{\langle\lambda\rangle^{2d}} \int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s-2\kappa}\chi_{\mathcal{S}_1}}{\langle\tau + \xi_2^2 - 2\xi\xi_2 + \xi^2\rangle^{2b}} d\xi_2.$$

Finally, since $s \le \kappa$, $b > 1/2$, and $d > 0$, we conclude that J_4 is bounded. □

IV. LOCAL EXISTENCE FOR LOW REGULARITY DATA

In this section, we prove, by using the Banach Fixed Point Theorem, the result of local well-posedness. We only show the case $0 < a < 1/2$ because the others follow the similar arguments.

Consider the following functional space where we will get our solution

$$\Sigma_\mu := \left\{(u,v) \in X^{\kappa,\frac{1}{2}+\mu} \times X_a^{s,\frac{1}{2}+\mu};\ \|u\|_{X^{\kappa,\frac{1}{2}+\mu}} \le M_1,\ \|v\|_{X_a^{s,\frac{1}{2}+\mu}} \le M_2\right\}, \tag{43}$$

where $0 < \mu \ll 1$ and $M_1, M_2 > 0$ will be chosen after.

We note that Σ_μ is a complete metric space with the standard norm

$$\|(u,v)\|_{\Sigma_\mu} := \|u\|_{X^{\kappa,\frac{1}{2}+\mu}} + \|v\|_{X_a^{s,\frac{1}{2}+\mu}}. \tag{44}$$

For $(u, v) \in \Sigma_\mu$, we define the maps

$$\Phi_1(u,v) = \psi_1(t) e^{it\partial_x^2} u_0 - i\psi_T(t) \int_0^t e^{i(t-t')\partial_x^2} \{\theta u(t') - (\overline{u}\cdot v)(t')\} dt', \tag{45}$$

$$\Phi_2(u,v) = \psi_1(t) e^{iat\partial_x^2} v_0 - i\psi_T(t) \int_0^t e^{ia(t-t')\partial_x^2} \left\{\alpha v(t') - \frac{a}{2}\left(u^2\right)(t')\right\} dt'. \tag{46}$$

We will choose $\mu < \mu(\kappa, s)$, where $d = \frac{1}{2} - 2\mu(\kappa,s)$ and $b = \frac{1}{2} + \mu(\kappa,s)$ satisfy the conditions of Propositions 1 and 2.

According to Lemma 1, with $b' = -d$ and Propositions 1 and 2, we have

$$\begin{aligned}
\|\Phi_1(u,v)\|_{X^{\kappa,\frac{1}{2}+\mu}} &\le c_0\|u_0\|_{H^\kappa} + c_1 T^\mu\left(\theta\|u\|_{X^{\kappa,-\frac{1}{2}+2\mu}} + \|\overline{u}v\|_{X^{\kappa,-\frac{1}{2}+2\mu}}\right)\\
&\le c_0\|u_0\|_{H^\kappa} + c_1 T^\mu\left(\theta\|u\|_{X^{\kappa,\frac{1}{2}+\mu}} + \|u\|_{X^{\kappa,\frac{1}{2}+\mu}}\|v\|_{X_a^{s,\frac{1}{2}+\mu}}\right)\\
&\le c_0\|u_0\|_{H^\kappa} + c_1 T^\mu\left(\theta M_1 + M_1 M_2\right),
\end{aligned}$$

$$\begin{aligned}
\|\Phi_2(u,v)\|_{X_a^{s,\frac{1}{2}+\mu}} &\le c_0\|v_0\|_{H^s} + c_2 T^\mu\left(\alpha\|v\|_{X_a^{s,-\frac{1}{2}+2\mu}} + \frac{a}{2}\left\|u^2\right\|_{X^{\kappa,-\frac{1}{2}+2\mu}}\right)\\
&\le c_0\|v_0\|_{H^s} + c_2 T^\mu\left(\alpha\|v\|_{X_a^{s,\frac{1}{2}+\mu}} + \frac{a}{2}\|u\|^2_{X^{\kappa,\frac{1}{2}+\mu}}\right)\\
&\le c_0\|v_0\|_{H^s} + c_2 T^\mu\left(\alpha M_2 + \frac{a}{2}M_1^2\right).
\end{aligned}$$

Defining $M_1 = 2c_0\|u_0\|_{H^k}$ and $M_2 = 2c_0\|v_0\|_{H^s}$, we have the following equations:

$$\|\Phi_1(u,v)\|_{X^{\kappa,\frac{1}{2}+\mu}} \le \frac{M_1}{2} + c_1 T^\mu\left(\theta M_1 + M_1 M_2\right)$$

and

$$\|\Phi_2(u,v)\|_{X_a^{s,\frac{1}{2}+\mu}} \le \frac{M_2}{2} + c_2 T^\mu\left(\alpha M_2 + \frac{a}{2}M_1^2\right).$$

Then $(\Phi_1(u,v), \Phi_2(u,v)) \in \Sigma_\mu$ for

$$T^\mu \le \frac{1}{2}\min\left\{\frac{1}{c_1(\theta + M_2)},\ \frac{M_2}{c_2(\alpha M_2 + \frac{a}{2}M_1^2)}\right\}. \tag{47}$$

Similarly, we have that

$$\|\Phi_1(u,v) - \Phi_1(\tilde{u},\tilde{v})\|_{X^{\kappa,\frac{1}{2}+\mu}} \le c_3(M_1,M_2) T^\mu\left(\|u-\tilde{u}\|_{X^{\kappa,\frac{1}{2}+\mu}} + \|v-\tilde{v}\|_{X_a^{s,\frac{1}{2}+\mu}}\right),$$

$$\|\Phi_2(u,v) - \Phi_2(\tilde{u},\tilde{v})\|_{X_a^{s,\frac{1}{2}+\mu}} \le c_4(M_1,M_2) T^\mu\left(\|u-\tilde{u}\|_{X^{\kappa,\frac{1}{2}+\mu}} + \|v-\tilde{v}\|_{X_a^{s,\frac{1}{2}+\mu}}\right).$$

Now, using (44) and inequalities above, we have

$$\left\|\Big(\Phi_1(u,v),\Phi_2(u,v)\Big)-\Big(\Phi_1(\tilde{u},\tilde{v}),\Phi_2(\tilde{u},\tilde{v})\Big)\right\|_{\Sigma_\mu}\leq\frac{1}{2}\|(u,v)-(\tilde{u},\tilde{v})\|_{\Sigma_\mu} \tag{48}$$

to

$$T^\mu\leq\frac{1}{4}\min\left\{\frac{1}{c_3(M_1,M_2)},\frac{1}{c_4(M_1,M_2)}\right\}.$$

Therefore, the map $\Phi_1\times\Phi_2{:}\Sigma_\mu\to\Sigma_\mu$ is a contraction, and by the Fixed Point Theorem there is a unique solution to the Cauchy problem for T satisfying (47) and (48).

□

Remark 4. The case $p=q=-1$ *can be treated by using the same ideas that in the case* $p=q=1$, *for any* $\sigma>0$.

Remark 5. The case $p=-1$ *and* $q=1$ *or* $p=1$ *and* $q=-1$ *(for all* $\sigma>0$*) is the same in the case* $p=q=1$ *for* $\sigma>2$.

V. GLOBAL WELL-POSEDNESS RESULTS

In this section, we will study the global well-posedness for system (49) below:

$$\begin{cases} i\partial_t u+p\partial_x^2 u-\theta u+\overline{u}v=0, & \\ i\sigma\partial_t v+q\partial_x^2 v-\alpha v+\frac{1}{2}u^2=0, & t\in[-T,T],\ x\in\mathbb{R},\\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x), & (u_0,v_0)\in H^\kappa(\mathbb{R})\times H^s(\mathbb{R}),\end{cases} \tag{49}$$

where u and v are complex valued functions.

One of the interests in working with system of equations in physics is to obtain stability for certain types of solutions. In this case, it is essential to have global well-posedness results.

Starting from the conservation law

$$E(u,v)(t)=\|u\|_{L^2}^2+2\sigma\|v\|_{L^2}^2, \tag{50}$$

it is known that if u and v are solutions of this system with initial conditions $(u_0,v_0)\in L^2\times L^2$, then $\forall t\in\mathbb{R}$, we have $E(u,v)(t)=E(u,v)(0)=\|u_0\|_{L^2}^2+2\sigma\|v_0\|_{L^2}^2$.

Our main result presented here is theorem 2.

To get the above result, we will follow the ideas presented in Refs. 4, 7, 16, and 8.

We note here that we did not explore the second quantity conserved for light regularities, for example, greater than 1, i.e.,

$$H(u,v)(t)=p\|u_x\|_{L^2}^2+q\|v_x\|_{L^2}^2+\theta\|u\|_{L^2}^2+\alpha\|v\|_{L^2}^2-\mathrm{Re}\langle u^2,\ \overline{v}\rangle_{L^2}. \tag{51}$$

A. Preliminary results

This section is devoted to the proof of the global well-posedness result stated in theorem 2 via the **I**-method.

Let $s\leq 0$ and $N>1$ be fixed. Let us define the Fourier multiplier operator

$$\widehat{I_N^{-s}u}(\xi)=\widehat{Iu}(\xi)=m(\xi)\widehat{u}(\xi),\ m(\xi)=\begin{cases}1, & |\xi|<N,\\ N^{-s}|\xi|^s, & |\xi|\geq 2N,\end{cases} \tag{52}$$

where m is a smooth non-negative function.

Lemma 3. The operator I *applies* $H^s(\mathbb{R})\longmapsto L^2$. *Moreover, the operator* I *commutes with differential operators and* $\overline{Iu}=I\overline{u}$. *That is,*

1. $\|I(u)\|_{L^2}\leq cN^{-s}\|u\|_{H^s}$,
2. $P(D)I(u)=I(P(D)u)$,

where P *is a polynomial and* $D=\dfrac{d}{idx}$ *is the differential operator.*

Proof. It follows from the definition of I and properties of the Fourier Transform. □

We will need the following.

Lemma 4 (Lemma 12.1 of Ref. 6). *Let $\alpha_0 > 0$ and $n \geq 1$. Suppose $Z, X_1, \ldots, X_n$ are translation-invariant Banach spaces and T is a translation invariant $n-$linear operator such that*

$$\left\| I_1^{\alpha} T(u_1, \ldots, u_n) \right\|_Z \leq c \prod_{j=1}^{n} \left\| I_1^{\alpha} u_j \right\|_{X_j},$$

for all $u_1, \ldots, u_n$, $0 \leq \alpha \leq \alpha_0$. Then,

$$\left\| I_N^{\alpha} T(u_1, \ldots, u_n) \right\|_Z \leq c \prod_{j=1}^{n} \left\| I_N^{\alpha} u_j \right\|_{X_j},$$

for all $u_1, \ldots, u_n$, $0 \leq \alpha \leq \alpha_0$, and $N \geq 1$. Here, the implied constant is independent of N.

Another essential result is

Lemma 5 (Lemma 5.1 of Ref. 8). *We have*

$$\left\| \left(D_x^{1/2} f \right) \cdot g \right\|_{L^2_{x,t}} \leq c \|f\|_{X^{0,1/2}} \|g\|_{X^{0,1/2}},$$

if $|\xi_2| \ll |\xi_1|$ for any $|\xi_1| \in \mathrm{supp}\left(\widehat{f}\right)$ and $|\xi_2| \in \mathrm{supp}(\widehat{g})$. Moreover, this estimate is true if f and/or g is replaced by its complex conjugate in the left-hand side of the inequality.

Remark 6. The lemma above is valid replacing $X^{0,1/2}$ by $X_a^{0,1/2}$.

B. Local well-posedness revisited

Now, we take $N \gg 1$ a sufficiently large integer and we denote by I the operator $I := I_N^{-s}$ for a given $s \in \mathbb{R}$.

We have that system (49) applied to the operator I is given by

$$\begin{cases} i\partial_t Iu + p\partial_x^2 Iu - \theta Iu + I(\bar{u}v) &= 0, \\ i\sigma\partial_t Iv + q\partial_x^2 Iv - \alpha Iv + \frac{1}{2} I\left(u^2\right) &= 0. \end{cases} \tag{53}$$

Let us state here a lemma that will be used to demonstrate the local well-posedness theorem and then re-obtain the bilinear estimates.

Lemma 6. Given $-1/2 < b' \leq b < 1/2$, $s \in \mathbb{R}$, $a \geq 0$, and $0 < T < 1$, the estimate below

$$\|\psi_T(t)u\|_{X_a^{s,b'}} \leq cT^{b-b'} \|u\|_{X_a^{s,b}} \tag{54}$$

holds.

Proof. See Ref. 10. □

Lemma 7. If $1/4 < d$, for $b_1, b_2 \in \mathbb{R}$ such that $(b_1, b_2) = \left(0, \frac{1}{2}+\right)$ or $(b_1, b_2) = \left(\frac{1}{2}+, 0\right)$, then

$$\|\bar{u} \cdot v\|_{X^{0,-d}} \leq c\|u\|_{X^{0,b_1}} \cdot \|v\|_{X_a^{0,b_2}}. \tag{55}$$

Proof. Without loss of generality, let us prove only the case $b_2 = 0$ and $b_1 = \frac{1}{2} +$. Following the ideas from Proposition 1, it follows that

$$\|\bar{u} \cdot v\|_{X^{0,-d}} \leq \|u\|_{X^{0,b_1}} \|v\|_{X^{0,b_1}} \left\| \frac{1}{\langle \tau_2 + a\xi_2^2 \rangle^{2b_2}} \int_{\mathbb{R}^2} \frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{2b_1} \langle \tau + \xi^2 \rangle^{2d}} d\xi d\tau \right\|_{L^\infty_{\xi_2,\tau_2}}.$$

On the right-hand side of the inequality above, using Lemma 16 and Lemma 17, we have that

$$\int_{\mathbb{R}^2} \frac{1}{\langle \tau_1 - \xi_1^2\rangle^{2b_1}\langle \tau + \xi^2\rangle^{2d}} d\xi d\tau \le \int_{\mathbb{R}^2} \frac{1}{\langle \tau_2 + 2\xi^2 + \xi_2^2 - 2\xi\xi_2\rangle^{2d}} d\xi d\tau \le c.$$

□

Analogously, we prove the lemma below.

Lemma 8. Consider $1/4 < d$. *Given* $b_1, b_2 \in \mathbb{R}$ *such that* $(b_1, b_2) = \left(0, \frac{1}{2}+\right)$ *or* $(b_1, b_2) = \left(\frac{1}{2}+, 0\right)$. *Then*

$$\|uw\|_{X_a^{0,-d}} \le c\|u\|_{X^{0,b_1}} \cdot \|w\|_{X^{0,b_2}}. \tag{56}$$

Remark 7. The above results are independent of the value of $a > 0$.

Now let us revisit the fixed-point theorem to find the best exponent for δ.

Proposition 7. For all $(u_0, v_0) \in H^s \times H^s$ *and* $s \ge -\frac{1}{4}$ *and* $0 < a < \frac{1}{2}$ *or* $s \ge -\frac{1}{2}$ *and* $a > \frac{1}{2}$, *system* (53) *has a unique local-in-time solution* $(u(t), v(t))$ *defined on the time interval* $[0, \delta]$ *for some* $\delta \le 1$ *satisfying*

$$\delta \sim \left(\|Iu_0\|_{L_x^2} + \|Iv_0\|_{L_x^2}\right)^{-\frac{4}{3}+}. \tag{57}$$

Furthermore, $\|Iu_0\|_{X^{0,1/2+}} + \|Iv_0\|_{X_a^{0,1/2+}} \le c(\|Iu_0\|_{L^2} + \|Iv_0\|_{L^2})$.

Proof. Using the Lemmas 3-8 the proof follows in a similar way to the Proposition 5.5 of Ref. 8. □

C. Almost conservation of the modified energy

Let us consider the energy E associated with the system (53)

$$E(Iu, Iv) = \|Iu\|_{L^2}^2 + 2\sigma\|Iv\|_{L^2}^2. \tag{58}$$

Theorem 3. *The functional energy* (58) *was derived with respect to the time given by*

$$\frac{d}{dt}E(Iu, Iv) = 2Im\left\{\int (I(\bar{u}v) - I\bar{u}Iv))I\bar{u}dx\right\} + 2Im\left\{\int \left(I(u^2) - (Iu)^2\right)I\bar{v}dx\right\}.$$

Proof. Also using the following fact $\displaystyle\int \bar{f}\cdot\partial_x^2 f = \int |\partial_x f|^2$, we get

$$\begin{aligned}\frac{d}{dt}E(Iu, Iv) &= \int \partial_t Iu \cdot I\bar{u} + \int Iu\cdot \partial_t I\bar{u} + 2\sigma\int \partial_t Iv\cdot I\bar{v} + 2\sigma\int Iv\cdot\partial_t I\bar{v} \\ &= -2\mathrm{Im}\left\{\int (I(\bar{u}v) - I\bar{u}Iv)\cdot I\bar{u}\right\} + 2\mathrm{Im}\left\{\int \left(I(\bar{u}^2) - (I\bar{u})^2\right)\cdot Iv\right\}.\end{aligned}$$

□

From now on $\delta = (\|Iu\|_{L^2} + \|Iv\|_{L^2})^{-4/3}$. Let us now estimate the modified energy. Using the fundamental theorem of calculus, we have

$$\begin{aligned}E(Iu, Iv)(\delta) - E(Iu, Iv)(0) &= 2\mathrm{Im}\int_0^\delta \left(\int (I(\bar{u}v) - I\bar{u}Iv))I\bar{u}dx\right)dt \\ &= 2\mathrm{Im}\int_0^\delta \left\langle (I(\bar{u}v) - I\bar{u}Iv)^\wedge;\ \widehat{Iu}\right\rangle_{L^2} dt \\ &\quad +2\mathrm{Im}\int_0^\delta \left\langle \left(I(u^2) - (Iu)^2\right)^\wedge;\ \widehat{Iv}\right\rangle_{L^2} dt.\end{aligned}$$

Observe that

$$\begin{aligned}(I(\overline{u}v) - I\overline{u}Iv)^{\wedge} &= m(\xi)\widehat{\overline{u}\cdot v} - \widehat{I\overline{u}} * \widehat{Iv}\\ &= \int \widehat{I\overline{u}}(\xi_1)\,\widehat{Iv}(\xi_2)\left(\frac{m(\xi)-m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)}\right)d\xi_1\end{aligned}$$

and

$$\begin{aligned}\left(I(u^2) - (Iu)^2\right)^{\wedge} &= m(\xi)\widehat{u^2} - \widehat{Iu} * \widehat{Iu}\\ &= \int \widehat{Iu}(\xi_1)\,\widehat{Iu}(\xi_2)\left(\frac{m(\xi)-m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)}\right)d\xi_1.\end{aligned}$$

Therefore,

$$\int_0^\delta \left\langle (I(\overline{u}v) - I\overline{u}Iv)^{\wedge};\ \widehat{Iu}\right\rangle_{L^2} dt = \int_0^\delta \int_{\mathbb{R}_\xi}\int_{\mathbb{R}_{\xi_1}} \widehat{I\overline{u}}(\xi_1)\,\widehat{Iv}(\xi_2)\,\widehat{I\overline{u}}(\xi)M(\xi,\xi_1)d\xi_1\,d\xi\,dt,$$

analogously, we have that

$$\int_0^\delta \left\langle \left(I(u^2) - (Iu)^2\right)^{\wedge};\ \widehat{Iv}\right\rangle_{L^2} dt = \int_0^\delta \int_{\mathbb{R}_\xi}\int_{\mathbb{R}_{\xi_1}} \widehat{Iu}(\xi_1)\,\widehat{Iu}(\xi_2)\,\widehat{I\overline{v}}(\xi)M(\xi,\xi_1)d\xi_1\,d\xi\,dt,$$

where $M(\xi,\xi_1) = \left(\dfrac{m(\xi)-m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)}\right)$.

We note that fixed $N > 1$, $|\xi_1| \sim N_1$, and $|\xi_2| \sim N_2$, we have

(i) If $2|\xi_1| \le |\xi_2|$ and $2|\xi_1| \le N$, then $|M(\xi,\xi_1)| \lesssim \frac{N_1}{N_2}$.
(ii) If $2|\xi_2| \le |\xi_1|$ and $2|\xi_2| \le N$, then $|M(\xi,\xi_1)| \lesssim \frac{N_2}{N_1}$.
(iii) If $2|\xi_1| \le |\xi_2|$ and $|\xi_1| \ge 2N$, then $|M(\xi,\xi_1)| \lesssim \frac{N_1}{N}$.
(iv) If $2|\xi_2| \le |\xi_1|$ and $|\xi_2| \ge 2N$, then $|M(\xi,\xi_1)| \lesssim \frac{N_2}{N}$.
(v) If $|\xi_1| \sim |\xi_2| \gtrsim N$, then $|M(\xi,\xi_1)| \lesssim \left(\frac{N_1}{N}\right)^2$.

By the symmetry of the variables, it is sufficient to verify only the statements (i), (iii), and (v).

We will use the fact that $m'(\xi) = -N|\xi|^{-2}$.

In the first case, as $|\xi_1| \ll N$, we get $m(\xi_1) = 1$, hence

$$|M(\xi,\xi_1)| = \left|\frac{m(\xi_1+\xi_2)-m(\xi_2)}{m(\xi_2)}\right| \sim \left|\frac{m'(\xi_2)|\xi_1|}{m(\xi_2)}\right| \lesssim \frac{N_1}{N_2}.$$

Still, to verify the item (iii), we observe that $\frac{1}{2}|\xi_2| \le |\xi_1+\xi_2| \le 2|\xi_2|$ and thereby,

$$\begin{aligned}\frac{m(\xi_1+\xi_2)-m(\xi_1)m(\xi_2)}{m(\xi_2)} &= \frac{N|\xi_1+\xi_2|^{-1} - N|\xi_2|^{-1}N|\xi_1|^{-1}}{N|\xi_2|^{-1}}\\ &= \frac{|\xi_2|}{|\xi_1+\xi_2|} - \frac{N}{|\xi_1|}\\ &\le 2 - \frac{N}{|\xi_1|} \sim 1.\end{aligned}$$

Then, (iii) follows easily from observation that $M(\xi,\xi_1) \sim \dfrac{1}{m(\xi_1)} = \dfrac{N_1}{N}$.

The last case follows from the fact that

$$\begin{aligned}m(\xi_1+\xi_2) - m(\xi_1)m(\xi_2) &= N|\xi_1+\xi_2|^{-1} - N^2|\xi_1|^{-1}|\xi_2|^{-1}\\ &\sim N\left(\frac{1}{2|\xi_1|} - \frac{N}{|\xi_1|^2}\right)\\ &= \frac{N}{2|\xi_1|}\frac{|\xi_1|-2N}{|\xi_1|} \sim 1.\end{aligned}$$

Therefore, $M(\xi,\xi_1)\sim\frac{1}{m(\xi_1)m(\xi_2)}\sim\left(\frac{N_1}{N}\right)^2$.

Considering

$$L_1 = 2\mathrm{Im}\int_0^\delta\int_{\mathbb{R}_\xi}\int_{\mathbb{R}_{\xi_1}}\widehat{I\overline{u}}(\xi_1)\,\widehat{Iv}(\xi_2)\,\widehat{I\overline{u}}(\xi)M(\xi,\xi_1)d\xi_1\,d\xi\,dt \tag{59}$$

and

$$L_2 = 2\mathrm{Im}\int_0^\delta\int_{\mathbb{R}_\xi}\int_{\mathbb{R}_{\xi_1}}\widehat{Iu}(\xi_1)\,\widehat{Iu}(\xi_2)\,\widehat{I\overline{v}}(\xi)M(\xi,\xi_1)d\xi_1\,d\xi\,dt, \tag{60}$$

we get

$$|E(Iu,Iv)(\delta)-E(Iu,Iv)(0)| = |L_1+L_2|.$$

Proposition 8. For $\sigma>2$ and $s\geq -1/2$, we have

$$|E(Iu,Iv)(\delta)-E(Iu,Iv)(0)|\leq N^{-\frac{1}{2}}\delta^{\frac{1}{2}}\|I(u)\|^2_{X^{0,\frac{1}{2}+}}\|I(v)\|_{X^{0,\frac{1}{2}+}}. \tag{61}$$

Proof. It is enough to estimate L_1 and L_2. We still note that L_1 and L_2 are equivalent. In this case, let us restrict ourselves to estimating L_1. Let us use the notation $|\xi| = |\xi_1+\xi_2|\sim N_3$

For $2|\xi_1|\leq|\xi_2|$ and $2|\xi_1|\leq N$ such that $|M(\xi,\xi_1)|\lesssim\frac{N_1}{N_2}$. Then, from Lemmas 5 and 6, we see that

$$\begin{aligned}|L_1| &\leq \left(\frac{N_1}{N_2}\right)^{1/2}\left\|D_x^{1/2}\widehat{I\overline{u}}(\xi_1)\cdot\widehat{Iv}(\xi_2)\right\|_{L^2}\left\|\widehat{I\overline{u}}\right\|_{L^2}\\ &\leq \left(\frac{N_1}{N_2}\right)^{1/2}N_3^{-1/2}\left\|\widehat{Iu}\right\|_{X^{0,1/2+}}\left\|\widehat{Iv}\right\|_{X^{0,1/2+}}\delta^{1/2}\left\|\widehat{Iu}\right\|_{X^{0,1/2+}}\\ &\leq N^{-1/2}\delta^{1/2}\|I(u)\|^2_{X^{0,\frac{1}{2}+}}\|I(v)\|_{X^{0,\frac{1}{2}+}}.\end{aligned}$$

The case (ii), that is, $2|\xi_2|\leq|\xi_1|$ and $2|\xi_2|\leq N$ follow by the symmetry of the variables. In the proof of cases (iii) and (iv), when $s=-1/2$ such that $|M(\xi,\xi_1)|\lesssim\left(\frac{N_1}{N}\right)^{1/2}$

$$\begin{aligned}|L_1| &\leq \left(\frac{N_1}{N}\right)^{1/2}\left\|D_x^{1/2}\widehat{I\overline{u}}(\xi_1)\cdot\widehat{Iv}(\xi_2)\right\|_{L^2}\left\|\widehat{I\overline{u}}\right\|_{L^2}\\ &\leq \left(\frac{N_1}{N}\right)^{1/2}N_3^{-1/2}\left\|\widehat{Iu}\right\|_{X^{0,1/2+}}\left\|\widehat{Iv}\right\|_{X^{0,1/2+}}\delta^{1/2}\left\|\widehat{Iu}\right\|_{X^{0,1/2+}}\\ &\leq N^{-1/2}\delta^{1/2}\|I(u)\|^2_{X^{0,\frac{1}{2}+}}\|I(v)\|_{X^{0,\frac{1}{2}+}}.\end{aligned}$$

For the last case, we have $|M(\xi,\xi_1)|\lesssim\frac{N_1}{N}$, when $|\xi_1|\sim|\xi_2|\gtrsim N$, thereby, $|\xi_1|\leq 2|\xi|$, and it implies

$$\begin{aligned}|L_1| &\leq \frac{N_1}{N}\left\|D_x^{1/2}\widehat{I\overline{u}}(\xi_1)\cdot\widehat{Iv}(\xi_2)\right\|_{L^2}\left\|\widehat{I\overline{u}}\right\|_{L^2}\\ &\leq \frac{N_1}{N}N_1^{-1/2}\left\|\widehat{Iu}\right\|_{X^{0,1/2+}}\left\|\widehat{Iv}\right\|_{X^{0,1/2+}}\delta^{1/2}\left\|\widehat{Iu}\right\|_{X^{0,1/2+}}\\ &\leq N^{-1}\delta^{1/2}\|I(u)\|^2_{X^{0,\frac{1}{2}+}}\|I(v)\|_{X^{0,\frac{1}{2}+}}.\end{aligned}$$

Since $|E(Iu,Iv)(\delta)-E(Iu,Iv)(0)| = |L_1+L_2|\leq|L_1|+|L_2|\leq c|L_1|$, we obtain

$$|E(Iu,Iv)(\delta)-E(Iu,Iv)(0)|\leq cN^{-1}\delta^{1/2}\|I(u)\|^2_{X^{0,\frac{1}{2}+}}\|I(v)\|_{X^{0,\frac{1}{2}+}}.$$

□

Following the same arguments presented above, we prove the following

Proposition 9. For $0<\sigma<2$ and $s\geq -1/4$, we have

$$|E(Iu,Iv)(\delta)-E(Iu,Iv)(0)|\leq N^{-\frac{1}{4}}\delta^{\frac{1}{2}}\|I(u)\|^2_{X^{0,\frac{1}{2}+}}\|I(v)\|_{X^{0,\frac{1}{2}+}}. \tag{62}$$

Proof. Analogous to the previous case. □

D. Global existence

In this subsection, we will demonstrate Theorem 2.

Proof. Given the initial conditions of the Cauchy problem (49) $(u_0, v_0) \in H^s \times H^s$ such that

$$\|I(u_0)\|_{L^2} \leq cN^{-s}\|\|_{H^s} \text{ and } \|I(v_0)\|_{L^2} \leq cN^{-s}\|v_0\|_{H^s}.$$

Applying the local well-posedness result of the Proposition 7, we see that there exists a unique solution in the time interval $[0, \delta]$, where $\delta \sim N^{-4s/3}$ and such that

$$\|I(u)\|_{X^{0,\frac{1}{2}+}} + \|I(v)\|_{X^{0,\frac{1}{2}+}} \leq cN^{-s}.$$

For $\sigma > 2$ and $s \geq -\frac{1}{2}$ and using the Proposition 8, we have

$$|E(Iu, Iv)(\delta) - E(Iu, Iv)(0)| \leq N^{-\frac{1}{2}}\delta^{\frac{1}{2}}N^{-3s}.$$

We should now prove that for every $T > 0$ we can extend our solution to the range $[0, T]$. In order to do it, it is enough to apply the local well-posedness Theorem7 until we reach this interval, that is, T/δ times. If the modified energy does not grow more than the initial one for this number of interactions, we can conclude that the result is extended up to the interval $[0, T]$, that is, we should have

$$|E(Iu, Iv)(\delta) - E(Iu, Iv)(0)|\frac{T}{\delta} \ll E(Iu_0, Iv_0). \tag{63}$$

Therefore, it is sufficient that

$$N^{-\frac{1}{2}}\delta^{\frac{1}{2}}N^{-3s}\frac{T}{\delta} \ll N^{-2s} \text{ or } N^{-\frac{1}{2}}\delta^{-\frac{1}{2}}N^{-3s}T \ll N^{-2s}. \tag{64}$$

Hence we conclude that $-\frac{1}{2} - 3s + \frac{2s}{3} \leq -2s$ because $\delta^{-1/2} \sim N^{2s/3}$. It turns out that for any $s \geq -1/2$ we can extend the solution at any time interval by taking $1 \ll N$.

The proof of the other case $(0 < \sigma < 2)$ follows similarly. □

The theorem showed in this section tells us that the solution of the Cauchy problem extends globally, in time, in the sense that it connects the points (0, 0) and (−1/2, −1/2) when $\sigma > 2$ and the points (0, 0) and (−1/4, −1/4) in the case $0 < \sigma < 2$.

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