ENAMA - Encontro Nacional de Análise Matemática e Aplicações
UEPB - Universidade Estadual da Paraíba e
UFCG - Universidade Federal de Campina Grande
XIV ENAMA - Novembro 2021 *87–88*

THE NONLINEAR QUADRATIC INTERACTIONS OF THE SCHRÖDINGER TYPE ON THE HALF-LINE

ISNALDO ISAAC BARBOSA[1] & MÁRCIO CAVALCANTE[2]

[1]IM, UFAL, Maceió (AL), Brazil, isnaldo@pos.mat.ufal.br,
[2]IM, UFAL, Maceió (AL), Brazil, marcio.melo@im.ufal.br

Abstract

In this work we study the initial boundary value problem associated with the coupled Schrödinger equations with quadratic nonlinearities, that appears in nonlinear optics, on the half-line. We obtain local well-posedness for data in Sobolev spaces with low regularity, by using a forcing problem on the full line with a presence of a forcing term in order to apply the Fourier restriction method of Bourgain. The crucial point in this work is the new bilinear estimates on the classical Bourgain spaces $X^{s,b}$ with $b < \frac{1}{2}$, jointly with bilinear estimates in adapted Bourgain spaces that will used to treat the traces of nonlinear part of the solution. Here the understanding of the dispersion relation is the key point in these estimates, where the set of regularity depends strongly of the constant a measures the scaling-diffraction magnitude indices.

This work was submitted for publication and can be accessed in https://arxiv.org/abs/2104.05137 .

1 Introduction

This work is dedicated to the study the initial boundary value problem associated to system nonlinear quadratic of the Schrodinger on the half-line, more precisely

$$\begin{cases} i\partial_t u(x,t) + \partial_x^2 u(x,t) + \bar{u}(x,t)v(x,t) = 0, & x \in (0,+\infty),\ t \in (0,T), \\ i\partial_t v(x,t) + a\partial_x^2 v(x,t) + u^2(x,t) = 0, & x \in (0,+\infty),\ t \in (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in (0,+\infty), \\ u(0,t) = f(t),\ v(0,t) = g(t), & t \in (0,T), \end{cases} \tag{1}$$

where u and v are complex valued functions, where $a > 0$. The model (1) is given by the nonlinear coupling of two dispersive equations of Schrödinger type through the quadratic terms $N_1(u,v) = \overline{u} \cdot v$ and $N_2(u,v) = u^2$.

An important point in this model is the fact that the functional mass is not conserved, since some bad terms of boundary appear in the mass functional. More precisely, define the functional of mass for the system (1) by

$$\mathcal{M}(t) = \|u(t)\|^2_{L^2_x(\mathbb{R}^+)} + \|v(t)\|^2_{L^2_x(\mathbb{R}^+)}.$$

Formally, by multiplying the first equation of the system (1) by $\overline{u}$ and the second equation by $\overline{v}$, integrating by parts, taking the imaginary part and using Im $(\overline{u}^2 v) = -\mathrm{Im}(u^2\overline{v})$, we get

$$\mathcal{M}(t) = \mathcal{M}(0) + \mathrm{Im}\int_0^t \overline{u}(0,s)\partial_x u(0,s)ds + a\mathrm{Im}\int_0^t \overline{v}(0,s)\partial_x v(0,s)ds. \tag{2}$$

This identity suggesters on the case of homogeneous boundary conditions a global result on the space $L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$.

Physically, according to the article [4], the complex functions u and v represent amplitude packets of the first and second harmonic of an optical wave. In the mathematical context on the paper [1] the first author obtained local well posedness for the model posed on real line by assuming low regularity assumptions.

2 Main Results

Our main local well-posedness result is the following statement.

Theorem 2.1. *Let the Sobolev index pair* (κ, s) *verifying* $s \neq \frac{1}{2}$ *and* $\kappa \neq \frac{1}{2}$ *and*

(i) $|\kappa| - 1/2 \leq s < \min\{\kappa + 1/2,\ 2\kappa + 1/2, 1\}$ *and* $\kappa < 1$ *for* $a > \frac{1}{2}$ *(first non resonant case);*

(ii) $0 \leq \kappa = s < 1$ *for* $a = \frac{1}{2}$ *(resonant case);*

(iii) $\max\{-\frac{1}{2}, |\kappa| - 1\} \leq s < \min\{\kappa + 1,\ 2\kappa + 1, 1\}$ *and* $\kappa < 1$ *for* $0 < a < \frac{1}{2}$ *(second non resonant case). For any* $a > 0$ *and* $(u_0, v_0) \in H^{\kappa}(\mathbb{R}^+) \times H^{s}(\mathbb{R}^+)$ *and* $(f, g) \in H^{\frac{2\kappa+1}{4}}(\mathbb{R}^+) \times H^{\frac{2s+1}{4}}(\mathbb{R}^+)$*, verifying the additional compatibility conditions*

$$\begin{cases} u(0) = f(0), & \text{for } \kappa > \frac{1}{2}; \\ v(0) = g(0), & \text{for } s > \frac{1}{2}. \end{cases} \tag{3}$$

Then there exist a positive time $T = T\left(\|u_0\|_{H^{\kappa}(\mathbb{R}^+)}, \|v_0\|_{H^{s}(\mathbb{R}^+)}, \|f\|_{H^{\frac{2\kappa+1}{4}}(\mathbb{R}^+)}, \|g\|_{H^{\frac{2s+1}{4}}(\mathbb{R}^+)}, a\right)$ *and a distributional solution* $(u(t), v(t))$ *for the initial boundary value problem (1) on the classes*

$$u \in C\left([0, T]; H^{\kappa}(\mathbb{R}^+)\right) \quad \text{and} \quad v \in C\left([0, T]; H^{s}(\mathbb{R}^+)\right). \tag{4}$$

Moreover, the map $(u_0, v_0) \longmapsto (u(t), v(t))$ *is locally Lipschitz from* $H^{\kappa}(\mathbb{R}^+) \times H^{s}(\mathbb{R}^+)$ *into* $C\left([0, T]; H^{\kappa}(\mathbb{R}^+) \times H^{s}(\mathbb{R}^+)\right)$*.*

The approach used to prove this result is based on the arguments introduced in [3] and [2]. The main idea to solve the IBVP (1) is the construction of an auxiliary forced IVP in the line $\mathbb{R}$, analogous to (1); more precisely:

$$\begin{cases} i\partial_t u(x,t) + \partial_x^2 u(x,t) + \bar{u}(x,t)v(x,t) = \mathcal{T}_1(x)h_1(t), & (x,t) \in \mathbb{R} \times (0,T) \\ i\partial_t v(x,t) + a\partial_x^2 v(x,t) + u^2(x,t) = \mathcal{T}_2(x)h_2(t), & (x,t) \in \mathbb{R} \times (0,T) \\ u(x,0) = \widetilde{u}_0(x), \quad v(x,0) = \widetilde{v}_0(x), & x \in \mathbb{R} \end{cases} \tag{5}$$

where $\mathcal{T}_1, \mathcal{T}_2$ are appropriate distributions supported in $\mathbb{R}^-$, $\tilde{u}_0, \tilde{v}_0$ are nice extensions of u_0 and v_0 in $\mathbb{R}$ and the boundary forcing functions h_1, h_2 are selected to ensure that

$$\widetilde{u}(0,t) = f(t) \quad \text{and} \quad \widetilde{v}(0,t) = g(t)$$

for all $t \in (0, T)$.

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